On the Existence of Stationary Points for the Steiglitz–McBride Algorithm

Phillip A. Regalia, Mamadou Mboup, and Mehdi Ashari

***Abstract*— Most convergence results for adaptive identification algorithms have been developed in sufficient order settings, involving an unknown system with known degree. Reduced-order settings, in which the degree of the unknown system is underestimated, are more common, but more difficult to analyze. Deducing stationary points in these cases typically involves solving nonlinear equations, hence the sparseness of results for reduced-order cases. If we allow ourselves the tractable case in which the input to an identification experiment is white noise, we pshall show that the Steiglitz–McBride method indeed admits a stationary point in reduced-order settings for which the resulting model is stable. Our interest in this study stems from a previous result, showing an attractive *a priori* bound on the mismodeling error at any such stationary point.**

***Index Terms*— Adaptive identification, reduced-order approximation, stationary points, Steiglitz–McBride algorithm.**

I. Introduction

Most convergence results in identification theory have been developed in sufficient order settings; the unknown system to be characterized is taken to have a rational transfer function whose degree is known or can be determined somehow. In practice, the unknown system may have a prohibitively large, if not infinite, degree. In such cases, identification by a model of finite degree—assumed inferior to that of the unknown system—is simply unattainable. The key query in such "reduced-order" settings, of course, is whether a candidate adaptation algorithm may nonetheless converge to a "good" approximation to the unknown system. For nongradient adaptation algorithms, convergence studies may no longer appeal to the analogy of seeking some minimum point. For adaptive IIR filters, in particular, convergence analyses are complicated by a nonlinear dependence of the filtered signals on the adaptive filter coefficients. Even deducing stationary points for (nongradient) adaptive IIR filters typically involves solving nonlinear equations, hence the sparseness of available results for reduced-order cases.

Here we address the existence of stationary points for the Steiglitz–McBride algorithm [1]–[3]. If we allow ourselves the tractable case in which the input sequence to an identification experiment is white noise, we shall prove that the Steiglitz–McBride algorithm always admits a stationary point for which the resulting model is stable, subject to a certain stability constraint on the (larger order) unknown system; whether or not the set of stationary points will always include an attractor point is not revealed from our analysis. Section I will motivate the problem under study and recall some previous results in this direction.

Consider a system identification setup in which the input $\{u(\cdot)\}$ is a discrete-time stochastic process, and the reference sequence is generated according to

$$y(n)=\sum_{k=0}^{\infty} h_k u(n-k)+\zeta(n).$$

Here $\{\zeta(\cdot)\}$ is an output disturbance which is assumed independent of the input sequence $\{u(\cdot)\}$. The unknown system has a transfer function given by

$$H(z)=\sum_{k=0}^{\infty} h_k z^k, \qquad |z|\le 1$$

in which z denotes the unit delay operator $zu(n)=u(n-1)$.

The Steiglitz–McBride algorithm adjusts the coefficients of a candidate rational transfer function

$$\hat{H}(z)=\frac{B(z)}{A(z)}=\frac{b_0+b_1z+\cdots+b_Mz^M}{1+a_1z+\cdots+a_Mz^M} \tag{1}$$

using the following adaptation algorithm [3].

Prefiltered Signals:

$$x(n)=-\sum_{k=1}^{M} a_k(n)x(n-k)+u(n) \tag{2}$$

$$\xi(n)=-\sum_{k=1}^{M} a_k(n)\xi(n-k)+y(n). \tag{3}$$

Output Error:

$$\hat{y}(n)=\sum_{k=0}^{M} b_k(n)x(n-k), \qquad e(n)=y(n)-\hat{y}(n). \tag{4}$$

Coefficient Updates:

$$b_k(n+1)=b_k(n)+\mu e(n)x(n-k), \qquad k=0,1,\cdots,M \tag{5}$$

$$a_k(n+1)=a_k(n)-\mu e(n)\xi(n-k), \qquad k=1,2,\cdots,M \tag{6}$$

with $\mu>0$ a small adaptation stepsize.

Convergence of this algorithm has been addressed by Fan [4] for slow adaptation. If the degree M chosen in (1) is the correct degree for the unknown system $H(z)$, this algorithm is (weakly) convergent to the correct system identification, provided the output disturbance $\{\zeta(\cdot)\}$ is white noise and the input $\{u(\cdot)\}$ is persistently exciting.

In reduced-order cases, convergence studies are more difficult. A first step in any such analysis is to isolate the stationary points in mean of the adaptation algorithm (5) and (6). If the input sequence $\{u(\cdot)\}$ is a stationary stochastic process, then the stationary points correspond to those parameter values $\{a_k\}$ and $\{b_k\}$ which, if held fixed, would result in the output error $e(n)$ being uncorrelated with the filtered regressors [5], i.e.,

$$E[x(n-k)e(n)]=0, \qquad k=0,1,\cdots,M \tag{7}$$

$$E[\xi(n-k)e(n)]=0, \qquad k=1,2,\cdots,M. \tag{8}$$

These same equations arise for the off-line variant studied in [2] and make sense, of course, only when $A(z)=1+a_1z+\cdots+a_Mz^M$ is minimum phase (no zeros in $|z|<1$). The left-hand sides of (7) and (8) show a nonlinear dependence on the coefficients $\{a_k\}$, thereby complicating a direct study of the solution set. Moreover, this system does not appear as the stationary point of some cost function, which bars an otherwise obvious proof of existence.

If existence can be ensured, then convergence can be studied in terms of the local stability properties about the stationary point (e.g.,

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P. A. Regalia is with the Département Signal et Image, Institut National des Télécommunications, F-91011 Evry cedex, France.

M. Mboup is with the UFR Mathématiques et Informatique, Université René Descartes, F-75270 Paris cedex 06, France.

M. Ashari was with the Laboratoire des Signaux et Systèmes, Ecole Supérieure d'Electricité, F-91192 Gif-sur-Yvette, France. He is now with the Microelectronics Group, Lucent Technologies, Allentown, PA 18103 USA.

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[4] and [2]). If, on the other hand, no solution exists, then in particular a stationary point cannot be approached; the convergence question would then seem void.

What is known about this system may be summarized as follows.

- If the output disturbance $\{\zeta(\cdot)\}$ is white noise, its presence makes no contribution to the system (7) and (8) and so does not bias the solution set [4], [2]. Nonetheless, one may always synthesize a stationary disturbance $\{\zeta(\cdot)\}$ such that, in arbitrarily poor signal-to-noise ratios, no solution to (8) exists for which $A(z)$ is a stable polynomial [6, Sec. 8.6]. To sidestep this shortcoming, we henceforth assume that $\{\zeta(\cdot)\}$ is a white noise process.
- In the sufficient order case [meaning the chosen filter order fulfills $M \geq \deg H(z)$] the choice $\hat{H}(z) = H(z)$ always solves (7) and (8) and becomes the sole solution if the input $\{u(\cdot)\}$ is persistently exciting [2], [4].
- For the reduced-order case, suppose the input $\{u(\cdot)\}$ is white noise as well. Whenever the system (7) plus (8) admits a solution for which $\hat{H}(z)$ is stable, the resulting mismodeling error $H(z) - \hat{H}(z)$ can be bounded in L_2 norm as [7]

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega}) - \hat{H}(e^{j\omega})|^2 \, d\omega\right)^{1/2} \leq \sigma_{M+1}$$

where σ_{M+1} is the $M+1$st singular value of the doubly infinite Hankel matrix

$$\Gamma_H \triangleq \begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This result asserts that, *if* a solution exists, the resulting approximation error is a well-behaved function of the degree of undermodeling, as measured by the Hankel singular values of $H(z)$. We should remark that other methods of identification theory, which can claim to converge correctly in the sufficient order case, do not always yield reliable models in reduced-order cases; see [8] for certain instrumental variable algorithms and [6, Sec. 9.7] for hyperstable algorithms such as (S)HARF.

Again, with the white noise assumption on $\{u(\cdot)\}$ and $\{\zeta(\cdot)\}$, we shall prove that (7) plus (8) indeed admits a solution for which the resulting model $\hat{H}(z)$ is stable, subject to a certain stability condition on the larger order system $H(z)$.

We recall that $H(z)$ is bounded-input/bounded-output (BIBO) stable if and only if its impulse response is absolutely summable

$$\sum_{k=0}^{\infty} |h_k| < \infty. \tag{9}$$

If we consider the derivative function

$$\frac{dH(z)}{dz} = \sum_{k=1}^{\infty} k h_k z^{k-1}$$

then this will remain BIBO stable provided $\Sigma_k \, k|h_k| < \infty$, a slightly stronger stability constraint than (9). We can continue this procedure up to the Mth derivative [where M is the chosen filter order in (1)]; if the resulting

$$\frac{d^M H(z)}{dz^M} = \sum_{k=M}^{\infty} k(k-1)\cdots(k-M+1) h_k z^{k-M}$$

remains BIBO stable, we shall say that $H(z)$ is "M-fold BIBO stable." Our main result is as follows.

Theorem 1: Suppose the input $\{u(\cdot)\}$ and disturbance $\{\zeta(\cdot)\}$ are both white noise processes. Then (7) plus (8) admits a solution for which the resulting $\hat{H}(z)$ is stable, whenever the larger order system $H(z)$ is M-fold BIBO stable.

Note that if the larger order system $H(z)$ is rational and BIBO stable, its Mth derivative $d^M H(z)/dz^M$ remains rational and BIBO stable. The M-fold BIBO stability criterion thus intervenes only for infinite-dimensional (or irrational) systems. If we consider the irrational transfer function

$$H(z) = \exp(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^k}{k!} + \cdots$$

then $d^M H(z)/dz^M = H(z)$, which remains BIBO stable. Conversely, the irrational transfer function

$$H(z) = 1 + z + \frac{z^2}{4} + \frac{z^3}{9} + \cdots + \frac{z^k}{k^2} + \cdots$$

is BIBO stable, but $dH(z)/dz$ is not. Thus the M-fold BIBO stability criterion accommodates some, though not all, infinite-dimensional BIBO systems.

Our result follows by judicious application of the Brouwer fixed-point theorem. Section II provides the initial machinery to set up our problem. Section III introduces the fixed-point theorem of interest and provides the technical construct to prove Theorem 1. Concluding remarks are synthesized in Section IV.

II. Problem Structure

We show in this section how the resolution of (7) plus (8) can be reduced to a fixed-point problem of a nonlinear map; our presentation is adapted from the thesis [9].

Let $f(z) = \Sigma_k \, f_k z^k$ and $g(z) = \Sigma_k \, g_k z^k$ be two transfer functions in L_2. If $\{u(\cdot)\}$ is a white noise process, we may equate

$$E[f(z)u(n) \cdot g(z)u(n)] = E[u^2(n)] \cdot \langle f(z), g(z) \rangle$$

in which

$$\langle f(z), g(z) \rangle = \sum_{k=-\infty}^{\infty} f_k g_k \tag{10}$$

denotes the standard inner product in L_2. To simplify notations to follow, we suppose the input process is scaled to unit variance: $E[u^2(n)] = 1$.

Now, if the filter coefficients $\{a_k\}$ and $\{b_k\}$ are held fixed, then the filtered signals from (2)–(4) may be expressed as

$$x(n-k) = \frac{z^k}{A(z)} u(n)$$

$$\xi(n-k) = \frac{z^k}{A(z)} [H(z)u(n) + \zeta(n)]$$

$$e(n) = [H(z) - \hat{H}(z)]u(n) + \zeta(n).$$

Assuming $\{\zeta(\cdot)\}$ is white, the orthogonality constraints (7) and (8) then become, respectively

$$\left\langle \begin{bmatrix} 1/A(z) \\ z/A(z) \\ \vdots \\ z^M/A(z) \end{bmatrix}, H(z) - \hat{H}(z) \right\rangle = \mathbf{0}_{M+1} \tag{11}$$

$$\left\langle \begin{bmatrix} z/A(z) \\ \vdots \\ z^M/A(z) \end{bmatrix} H(z), H(z) - \hat{H}(z) \right\rangle = \mathbf{0}_M \tag{12}$$

in which the inner products act componentwise in each expression. For a given $H(z)$, we examine existence of a stable $\hat{H}(z) = B(z)/A(z)$ which is compatible with these constraints.

Introduce now the all-pass function

$$\begin{aligned} V(z) &\triangleq \frac{z^M A(z^{-1})}{A(z)} \\ &= \frac{a_M + a_{M-1}z + \cdots + a_1 z^{M-1} + z^M}{1 + a_1 z + \cdots + a_{M-1}z^{M-1} + a_M z^M} \\ &= \sum_{k=0}^{\infty} v_k z^k \end{aligned} \tag{13}$$

with $\{v_k\}$ its impulse response. One may show [10], [11], [6] that (11) is satisfied if and only if

$$H(z) - \hat{H}(z) = V(z)\sum_{k=1}^{\infty} g_k z^k = V(z)g(z) \tag{14}$$

with $g(z)$ strictly causal, i.e., $H(z) - \hat{H}(z)$ has zeros coinciding with those of $V(z)$, plus a zero at $z = 0$ from $g(0) = 0$. For any minimum phase $A(z)$, this constraint reduces to a linear system in the coefficients $\{b_k\}$, which is always solvable [6], [10], [11].

Since $V(z)$ is all-pass, we have $V(z^{-1})V(z) = 1$, in which $V(z^{-1}) = \Sigma_{k=0}^{\infty} v_k z^{-k}$ is anti-causal. Then (14) can be rearranged as

$$V(z^{-1})H(z) - V(z^{-1})\hat{H}(z) = g(z).$$

It is easy to check that $V(z^{-1})\hat{H}(z) = [z^{-M}B(z)]/A(z^{-1})$ is an anti-causal transfer function, so that $g(z)$ is the strictly causal part of the product $V(z^{-1})H(z)$. The impulse response coefficients $\{g_k\}$ then relate to the impulse responses $\{h_k\}$ and $\{v_k\}$ according to

$$\underbrace{\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \end{bmatrix}}_{\boldsymbol{g}} = \underbrace{\begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\Gamma_H} \underbrace{\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{bmatrix}}_{\boldsymbol{v}}. \tag{15}$$

Upon substituting $H(z) - \hat{H}(z) = V(z)g(z)$ from (14) into the second system of (12), we obtain

$$\begin{aligned} \mathbf{0}_M &= \left\langle \begin{bmatrix} z/A(z) \\ \vdots \\ z^M/A(z) \end{bmatrix} H(z), V(z)g(z) \right\rangle \\ &= \left\langle \begin{bmatrix} z/A(z) \\ \vdots \\ z^M/A(z) \end{bmatrix} V(z^{-1}), H(z^{-1})g(z) \right\rangle \\ &= \left\langle \begin{bmatrix} z^{-1}/A(z^{-1}) \\ \vdots \\ z^{-M}/A(z^{-1}) \end{bmatrix} V(z), H(z)g(z^{-1}) \right\rangle. \end{aligned} \tag{16}$$

Each term in the left-hand operand of the final inner product is a causal function, as can be seen by

$$\frac{z^{-k}}{A(z^{-1})}V(z) = \frac{z^{-k}}{A(z^{-1})}\frac{z^M A(z^{-1})}{A(z)} = \frac{z^{M-k}}{A(z)}.$$

In this way, the orthogonality constraint (16) may be rewritten as

$$\mathbf{0}_M = \left\langle \begin{bmatrix} 1/A(z) \\ z/A(z) \\ \vdots \\ z^{M-1}/A(z) \end{bmatrix}, H(z)g(z^{-1}) \right\rangle. \tag{17}$$

Now, by Parseval's relation [cf. (10)], only the causal part of the right-hand operand in (17) intervenes. If we let $f_0 + f_1 z + f_2 z^2 + \cdots$ denote the causal part of the product $H(z)g(z^{-1})$, then equating like powers of z leads to the system

$$\underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}}_{\boldsymbol{f}} = \underbrace{\begin{bmatrix} h_1 & h_2 & h_3 & \cdots \\ h_2 & h_3 & h_4 & \cdots \\ h_3 & h_4 & h_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\Gamma_H} \underbrace{\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \end{bmatrix}}_{\boldsymbol{g}} = \Gamma_H^2 \boldsymbol{v} \tag{18}$$

since $\boldsymbol{g} = \Gamma_H \boldsymbol{v}$ in (15).

Expand now the all-pole function $1/A(z)$ in terms of its impulse response as

$$\frac{1}{A(z)} \triangleq C(z) = c_0 + c_1 z + c_2 z^2 + \cdots, \qquad |z| < 1 \tag{19}$$

and arrange the coefficients as a column vector $\boldsymbol{c} = [c_0 \quad c_1 \quad c_2 \quad \cdots]^t$. Define now $\mathcal{Z}$ as the down-shift matrix with ones on the subdiagonal and zeros elsewhere, such that $\mathcal{Z}\boldsymbol{c} = [0 \quad c_0 \quad c_1 \quad \cdots]^t$ is the impulse response vector from $z/A(z)$, and so on. Writing (17) in terms of impulse responses then gives

$$\mathbf{0}_M = \begin{bmatrix} \boldsymbol{c}^t \\ \boldsymbol{c}^t \mathcal{Z}^t \\ \vdots \\ \boldsymbol{c}^t(\mathcal{Z}^t)^{M-1} \end{bmatrix} \boldsymbol{f} = \begin{bmatrix} \boldsymbol{c}^t \\ \boldsymbol{c}^t \mathcal{Z}^t \\ \vdots \\ \boldsymbol{c}^t(\mathcal{Z}^t)^{M-1} \end{bmatrix} \Gamma_H^2 \boldsymbol{v} \tag{20}$$

in which we substitute $\boldsymbol{f}$ from (18) in the latter equality.

From (13) for $V(z)$ finally, the impulse response vector $\boldsymbol{v}$ may be written as

$$\boldsymbol{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{bmatrix} = [\boldsymbol{c} \quad \mathcal{Z}\boldsymbol{c} \quad \cdots \quad \mathcal{Z}^M\boldsymbol{c}] \begin{bmatrix} a_M \\ \vdots \\ a_1 \\ 1 \end{bmatrix}.$$

With this, (20) becomes

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \times \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{c}^t \\ \boldsymbol{c}^t \mathcal{Z}^t \\ \vdots \\ \boldsymbol{c}^t(\mathcal{Z}^t)^{M} \end{bmatrix} \Gamma_H^2 [\boldsymbol{c} \quad \mathcal{Z}\boldsymbol{c} \quad \cdots \quad \mathcal{Z}^M\boldsymbol{c}]}_{\triangleq \boldsymbol{P}(\boldsymbol{a})} \begin{bmatrix} a_M \\ \vdots \\ a_1 \\ 1 \end{bmatrix} \tag{21}$$

in which the left-most matrix on the right-hand side has been augmented by one row in order to obtain a symmetric matrix $\boldsymbol{P}(\boldsymbol{a})$ in the middle. Since this matrix varies with $\boldsymbol{a} = [a_1, \cdots, a_M]$, this system is nonlinear in the coefficients $\{a_k\}$.

Introduce now dummy parameters $\alpha_1, \cdots, \alpha_M$. We shall verify shortly that $\boldsymbol{P}(\boldsymbol{a})$ is positive definite; this will allow us to solve

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \times \end{bmatrix} = \boldsymbol{P}(\boldsymbol{a}) \begin{bmatrix} \alpha_M \\ \vdots \\ \alpha_1 \\ 1 \end{bmatrix} \tag{22}$$

for the coefficients $\alpha = [\alpha_1, \cdots, \alpha_M]$. The solution α varies, of course, with the coefficients $\boldsymbol{a}$, thereby defining a function of $\boldsymbol{a}$: $\alpha = \mathcal{F}(\boldsymbol{a})$. System (21) then admits a solution $\boldsymbol{a}$ if and only if the function $\mathcal{F}$ implicitly defined by (22) admits a fixed point $\alpha = \mathcal{F}(\boldsymbol{a}) = \boldsymbol{a}$.

III. Fixed-Point Theorem

We begin this section with the Brouwer fixed-point theorem [12, p. 45], [13, p. 84].

Theorem 2: Let $\mathcal{D}$ be a closed, bounded, and convex domain, and let $\mathcal{F}(\cdot)$ be any continuous function which maps $\mathcal{D}$ into $\mathcal{D}$. Then $\mathcal{F}(\cdot)$ admits a fixed point in $\mathcal{D}$.

Return now to the polynomial $A(z) = 1 + a_1 z + \cdots + a_M z^M$. By the Schur/Levinson recursion, this polynomial may be mapped to its reflection coefficients κ_i^A, and $A(z)$ will be strictly minimum phase if and only if

$$|\kappa_i^A| < 1, \qquad i = 1, 2, \cdots, M. \tag{23}$$

This region is clearly bounded and convex in the reflection coefficient space, but open.

Introduce its closed counterpart as

$$\mathcal{D} = \{\kappa_i^A : |\kappa_i^A| \le 1 \text{ for all } i\}.$$

Letting $\partial\mathcal{D}$ denote the boundary, i.e., points where one or more reflection coefficients reach unit magnitude, then the open domain from (23) becomes $\mathcal{D}$ minus its boundary: $\mathcal{D} \ominus \partial\mathcal{D}$.

Now, for all κ_i^A in the open domain (23), the coefficients $\{a_k\}$ vary continuously with κ_i^A. The matrix $\boldsymbol{P}(\boldsymbol{a})$ is a continuous function of $\boldsymbol{a}$, and the solution α from (22) is continuous in $\boldsymbol{P}(\boldsymbol{a})$. Whenever the resulting polynomial $\alpha(z) = 1 + \alpha_1 z + \cdots + \alpha_M z^M$ is minimum phase (to be addressed shortly), it may be mapped into its reflection coefficients κ_i^α in a continuous manner. We may thus think of $\mathcal{F}(\cdot)$ as mapping the reflection coefficients κ_i^A continuously to the reflection coefficients κ_i^α

$$\underbrace{[\kappa_1^\alpha, \cdots, \kappa_M^\alpha]}_{\kappa^\alpha} = \underbrace{[\mathcal{F}_1(\kappa^A), \cdots, \mathcal{F}_M(\kappa^A)]}_{\mathcal{F}(\kappa^A)}.$$

Our proof proceeds in two steps.

1) We show first that whenever the reflection coefficients $\{\kappa_i^A\}$ lie in the open hypercube (23), so do the resulting reflection coefficients κ_i^α obtained from $\alpha(z)$

$$\text{for all } \kappa^A \in \mathcal{D} \ominus \partial\mathcal{D}, \qquad \kappa^\alpha = \mathcal{F}(\kappa^A) \in \mathcal{D} \ominus \partial\mathcal{D}.$$

2) We then show that, as κ^A reaches any boundary point on $\partial\mathcal{D}$, the resulting $\kappa^\alpha = \mathcal{F}(\kappa^A)$ remains in $\mathcal{D} \ominus \partial\mathcal{D}$, whenever $H(z)$ is M-fold BIBO stable.

The conditions of Theorem 2 will then be satisfied so that the map $\kappa^\alpha = \mathcal{F}(\kappa^A)$ will admit a fixed point in the open domain (23). This will give $A(z)$ strictly minimum phase and thus $\hat{H}(z) = B(z)/A(z)$ as a stable transfer function.

Part 1 will follow upon exposing the structure of the $\boldsymbol{P}$ matrix and then invoking a result by Mullis and Roberts [14]. To begin, introduce a "square root" to the matrix $\boldsymbol{P}$ from (21)

$$\boldsymbol{S} \triangleq \Gamma_H[\boldsymbol{c} \quad \mathcal{Z}\boldsymbol{c} \quad \cdots \quad \mathcal{Z}^M\boldsymbol{c}]. \tag{24}$$

This gives $\boldsymbol{P} = \boldsymbol{S}^t\boldsymbol{S}$. By Kronecker's theorem (e.g., [11]), $\operatorname{rank}(\Gamma_H) = \deg H(z)$, which we assume greater than M, i.e., $\deg H(z)$ is underestimated. We show first that $\boldsymbol{S}$ has full rank $M+1$, so that $\boldsymbol{P}$ will be positive definite.

If $\boldsymbol{S}$ did not have full rank, then some linear combination of the column vectors $\boldsymbol{c}, \mathcal{Z}\boldsymbol{c}, \cdots, \mathcal{Z}^M\boldsymbol{c}$ would lie in the null space of Γ_H. But the z-transform of any null vector of Γ_H must contain an all-pass factor of degree equal to $\operatorname{rank}[\Gamma_H] > M$ (e.g., [6]). The z-transform of any linear combination of $\boldsymbol{c}, \cdots, \mathcal{Z}^M\boldsymbol{c}$, though, yields a rational function of degree M or less. Thus $\boldsymbol{S}$ has full rank.

Now let $\boldsymbol{s} = [s_1 \quad s_2 \quad s_3 \quad \cdots]^t = \Gamma_H\boldsymbol{c}$ denote the first column of $\boldsymbol{S}$. Since Γ_H is a doubly infinite Hankel matrix, it satisfies (by definition of Hankel) $\Gamma_H\mathcal{Z} = \mathcal{Z}^t\Gamma_H$. Accordingly, $\boldsymbol{S}$ is also a Hankel matrix

$$\begin{aligned}\boldsymbol{S} &= \Gamma_H[\boldsymbol{c} \quad \mathcal{Z}\boldsymbol{c} \quad \cdots \quad \mathcal{Z}^M\boldsymbol{c}] \\ &= [\boldsymbol{s} \quad \mathcal{Z}^t\boldsymbol{s} \quad \cdots \quad (\mathcal{Z}^t)^M\boldsymbol{s}] \\ &= \begin{bmatrix} s_1 & s_2 & \cdots & s_{M+1} \\ s_2 & s_3 & \cdots & s_{M+2} \\ s_3 & s_4 & \cdots & s_{M+3} \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix} \quad [\infty \times (M+1)].\end{aligned} \tag{25}$$

Extend now this matrix into the form

$$\left[\begin{array}{cccc} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & s_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & s_1 & \cdots & s_M \\ \hline s_1 & s_2 & \cdots & s_{M+1} \\ s_2 & s_3 & \cdots & s_{M+2} \\ \vdots & \vdots & \ddots & \vdots \end{array}\right] \begin{array}{c} \\ \\ \\ \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \boldsymbol{S} \end{array}$$

while maintaining the Hankel structure. One may check that the gramian of this matrix is symmetric and Toeplitz, such that the gramian of $\boldsymbol{S}$ may be written as

$$\boldsymbol{P} = \boldsymbol{S}^t\boldsymbol{S} = \begin{bmatrix} r_0 & r_1 & \cdots & r_M \\ r_1 & r_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_1 \\ r_M & \cdots & r_1 & r_0 \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & s_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & s_1 & \cdots & s_M \end{bmatrix}^2 \tag{26}$$

where

$$r_k = \sum_{i=1}^{\infty} s_i s_{i+k}, \qquad k = 0, 1, \cdots, M.$$

Careful inspection shows that $\boldsymbol{P}$ assumes precisely the structure studied by Mullis and Roberts in [14], who showed that the solution to the normal equations (22) will yield $\alpha(z) = 1 + \alpha_1 z + \cdots + \alpha_M z^M$ as a strictly minimum phase polynomial whenever this structured matrix $\boldsymbol{P}$ is positive definite. This gives Part 1 of the proof.

For Part 2, we now examine the behavior of the polynomial $\alpha(z)$ in the limit as one or more reflection coefficients κ_i^A reach unit magnitude, corresponding to $A(z)$ having zeros on the unit circle. Note that the causal expansion (19) remains valid as zeros of $A(z)$ reach the unit circle, but the region of convergence then excludes the unit circle. The rate of growth in the sequence $\{c_k\}$ is determined by the multiplicities of any unit-circle zeros of $A(z)$. In the worst case, where $A(z)$ has a zero of multiplicity M on the unit circle, one may readily bound the growth as

$$|c_k| \le \beta \cdot (k+1)^{M-1}, \qquad \text{for all } k \tag{27}$$

for some constant β.

We shall show that $\boldsymbol{P}$ nonetheless remains bounded at any boundary point $\partial\mathcal{D}$, subject to the M-fold BIBO stability contraint; that $\boldsymbol{P}$ may be written as a structured positive definite matrix as per (26) will follow by the same arguments above, so that the resulting $\alpha(z)$ will remain strictly minimum phase.

Now, the (i, j) entry of $\boldsymbol{P}$ (counting from zero) appears as $\boldsymbol{P}_{ij} = \boldsymbol{s}^t \mathcal{Z}^i (\mathcal{Z}^t)^j \boldsymbol{s}$. As $||\mathcal{Z}|| = 1$, the Cauchy–Schwartz inequality gives

$$|\boldsymbol{P}_{ij}| \leq ||(\mathcal{Z}^t)^i \boldsymbol{s}|| \cdot ||(\mathcal{Z}^t)^j \boldsymbol{s}|| \leq ||\boldsymbol{s}||^2 = \sum_{k=1}^{\infty} s_k^2 = \boldsymbol{P}_{00}$$

so that it suffices to examine the leading entry $\boldsymbol{P}_{00}$. If we show that $\{s_k\}$ is absolutely summable $(\Sigma_k |s_k| < \infty)$ whenever $H(z)$ is M-fold BIBO stable, square summability will follow as well.

Now, using the bound (27), the system $\boldsymbol{s} = \Gamma_H \boldsymbol{c}$ from the first column of (25) can be bounded componentwise as

$$\begin{aligned}
|s_1| &\leq \beta(|h_1| + 2^{M-1}|h_2| + 3^{M-1}|h_3| + 4^{M-1}|h_4| + \cdots) \\
|s_2| &\leq \beta(\qquad\quad |h_2| + 2^{M-1}|h_3| + 3^{M-1}|h_4| + \cdots) \\
|s_3| &\leq \beta(\qquad\qquad\qquad\quad |h_3| + 2^{M-1}|h_4| + \cdots) \\
|s_4| &\leq \beta(\qquad\qquad\qquad\qquad\qquad\quad |h_4| + \cdots) \\
&\ddots
\end{aligned}$$

The absolute sum of $\{s_k\}$ is then bounded as

$$\sum_{k=1}^{\infty} |s_k| \leq \beta \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} i^{M-1} \right) |h_k|$$

where $\Sigma_{i=1}^k i^{M-1} \leq k^M$. This gives

$$\sum_{k=1}^{\infty} |s_k| \leq \beta \sum_{k=1}^{\infty} k^M |h_k|$$

in which the right-hand side remains finite whenever the functions $H(z), dH(z)/dz, \cdots, d^M H(z)/dz^M$ remain BIBO stable. This completes the boundary case, to prove Theorem 1.

IV. Concluding Remarks

Our main result shows that the Steiglitz–McBride algorithm does admit a stationary point in reduced-order cases, though the present result is restricted to white noise inputs and disturbances. Our interest in affirming existence stems from a previous result [7] showing an attractive error bound at any such stationary point. Whether the set of stationary points will always include an attractor point to either the on-line [3] or off-line [2] version is not revealed from our analysis.

One is tempted, of course, to extend this result to correlated inputs; let us pinpoint where the construct breaks down. For correlated inputs, one may still arrive at a matrix equation akin to (22), but the resulting $\boldsymbol{P}$ matrix is no longer close to Toeplitz in the sense of (26), i.e., its displacement rank increases. The coefficients $\{\alpha_k\}$ then need not yield the coefficients of a minimum phase polynomial (see e.g., [2]). The fixed-point theorem evoked here no longer applies, but this, of course, does not belie the existence of a stationary point for that case.

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Calculation of the Structured Singular Value with Gradient-Based Optimization Algorithms on a Lie Group of Structured Unitary Matrices

Jeroen Dehaene, Cheng Yi, and Bart De Moor

***Abstract*—The structured singular value problem, which is a basic problem in robustness analysis and design of multivariable controllers, can be formulated as an optimization problem over the manifold of unitary matrices with a given structure. We show how geometric optimization methods, such as the steepest ascent method and the conjugate gradient method for optimization on a Riemannian manifold, lead to algorithms giving a guaranteed nontrivial lower bound for the structured singular value.**

***Index Terms*—Gradient-based optimization, structured singular value (μ).**

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J. Dehaene and B. De Moor are with ESAT-SISTA, K. U. Leuven, B-3001 Leuven (Heverlee), Belgium.

C. Yi is with Dynamic Control Systems Inc., Delta, B. C. V4G 1H5 Canada.

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