DESIGN OF BIORTHOGONAL MULTI-BAND FIR FILTER BANKS GIVEN SEVERAL OF THE ANALYSIS AND SYNTHESIS FILTERS

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ABSTRACT

The problem of designing an N-band biorthogonal FIR analysis/synthesis system, given K of its analysis and synthesis filters, is addressed in this paper. A necessary and sufficient condition on the K filters, that guides their selection, is derived and the solution set is completely parameterized via ladder-type structures that guarantee the perfect reconstruction property under both coefficient quantization and roundoff errors. Results are presented for both nonlinear- and linear-phase systems. A design example is provided to illustrate the theory.

1 INTRODUCTION

Designing an N-band analysis/synthesis perfect reconstruction (PR) FIR filter bank (FB) given $K \geq 1$ of the analysis filters constitutes a powerful means of constructing scalar- and vector-valued wavelet bases with desired properties in their analysis functions, and has also been shown to facilitate the design and realization of PRFB's. A complete solution to this so-called *analysis (N, K)-problem* has been reported both for the paraunitary [7, 16] and the general biorthogonal linear-phase (LP) and non-LP cases [7, 2]. However, such a design approach does not provide any direct control on the quality of the synthesis FB. Of course, this is of no concern in the case of orthogonal analysis/synthesis systems (ASS) where the synthesis filters are simple time-reversed versions of the analysis ones [12]. For nonorthogonal PR ASS however, this becomes an important issue, especially in applications like wavelet design, where it is well-known that the regularity of the analysis and synthesis wavelets has to be separately enforced [5]. Furthermore, it has been pointed out (e.g., [3]) that the regularity of the synthesis wavelets may be equally or even more important than that of the analysis ones. Moreover, the interest in considering *biorthogonal* multiband wavelets was demonstrated in [5], among others.

Hence the question arises of whether one can build an N-band biorthogonal ASS with $1 \leq K < N$ filters being selected a-priori in *both* the analysis and synthesis banks. This problem was posed in [5] in the context of N-band wavelet design,[1] though no solution was provided. A complete solution to the above problem (to be referred to hereafter as the *analysis/synthesis (N, K)-problem*) was reported in [8]. Nevertheless, the problems of designing the K analysis/synthesis filters and optimizing the frequency resolution of the resulting ASS need to be studied. In this paper, the results of [8] are revisited, emphasizing aspects of the design phase. Robust realization structures are elaborated. The filter selection problem is investigated and shown to reduce to the factorization of appropriately designed Nth-band filters. The theory presented is illustrated via a design example.

2 THE ANALYSIS/SYNTHESIS (N, K)-PROBLEM

Let $\{H_i(z)\}_{i=0}^{N-1}$ and $\{G_i(z)\}_{i=0}^{N-1}$ denote the analysis and synthesis FB's, respectively, in an N-band ASS.[2] A (essentially) sufficient and necessary condition for this ASS to be PR is that

$$\boldsymbol{G}_p^T(z)\boldsymbol{H}_p(z) = z^{-k}\boldsymbol{I}, \quad k \text{ a nonnegative integer} \tag{1}$$

with $\boldsymbol{H}_p(z) = [H_{i,j}(z)]$ and $\boldsymbol{G}_p(z) = [G_{i,j}(z)]$ denoting the matrices of the (type-I) analysis and (type-II) synthesis polyphase components, respectively [14]. Thus, for an FIR ASS to be PR, the matrix $\boldsymbol{H}_p(z)$ (as well as $\boldsymbol{G}_p(z)$) has to be unimodular, i.e., $\det(\boldsymbol{H}_p(z)) = z^{-l}$ for some nonnegative integer l [14]. Define the $K \times N$ matrices $\boldsymbol{h}_0^T(z)$ and $\boldsymbol{g}_0^T(z)$ as

$$\boldsymbol{h}_0^T(z) = \begin{bmatrix} H_{0,0}(z) & \cdots & H_{0,N-1}(z) \\ \vdots & \ddots & \vdots \\ H_{K-1,0}(z) & \cdots & H_{K-1,N-1}(z) \end{bmatrix} \tag{2}$$

and similarly for $\boldsymbol{g}_0^T(z)$, such that they satisfy

$$\boldsymbol{h}_0^T(z)\boldsymbol{g}_0(z) = z^{-k}\boldsymbol{I}_K. \tag{3}$$

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[1] The related problem of designing biorthogonal wavelet bases starting from any two multiresolution analyses was recently studied for the two-band case [1].

[2] Only real FB's will be considered.

We may then state the analysis/synthesis (N,K)-problem as that of completing these two matrices up to $N \times N$ unimodular matrices $\boldsymbol{H}_p(z)$ and $\boldsymbol{G}_p(z)$, respectively, satisfying (1). As shown in [7], a Smith decomposition of $\boldsymbol{h}_0^T(z)$ provides us with a unimodular matrix $\boldsymbol{B}(z)$ such that $\boldsymbol{h}_0^T(z)\boldsymbol{B}(z) = \begin{bmatrix} \boldsymbol{I}_K & \boldsymbol{0} \end{bmatrix}$, whose inverse yields a particular solution to the corresponding analysis (N,K)-problem:

$$\hat{\boldsymbol{H}}_p(z) \triangleq \begin{bmatrix} \boldsymbol{h}_0^T(z) \\ \hat{\boldsymbol{\mathcal{H}}}(z) \end{bmatrix} = \boldsymbol{B}^{-1}(z). \qquad (4)$$

The general solution can be parameterized in terms of a particular one as follows [7]:

Theorem 1 *The set of valid analysis polyphase matrices is generated as:*

$$\boldsymbol{H}_p(z) = \begin{bmatrix} \boldsymbol{I}_K & \boldsymbol{0} \\ \boldsymbol{E}(z) & \boldsymbol{U}(z) \end{bmatrix} \hat{\boldsymbol{H}}_p(z), \qquad (5)$$

where $\boldsymbol{U}(z)$ is any $(N-K)\times(N-K)$ unimodular matrix and $\boldsymbol{E}(z)$ ranges over all $(N-K)\times K$ polynomial matrices.

A particular solution for the synthesis polyphase matrix is given by $\hat{\boldsymbol{G}}_p^T(z) = \boldsymbol{B}(z)$. From (1) and (5) one may express the *general* synthesis polyphase matrix in terms of a particular one as follows:

$$\boldsymbol{G}_p^T(z) = z^{-k}\hat{\boldsymbol{G}}_p^T(z) \begin{bmatrix} \boldsymbol{I}_K & \boldsymbol{0} \\ -\boldsymbol{U}^{-1}(z)\boldsymbol{E}(z) & \boldsymbol{U}^{-1}(z) \end{bmatrix} \qquad (6)$$

We look then for the appropriate choice of the parameters $\boldsymbol{E}(z)$ and $\boldsymbol{U}(z)$ that makes $\boldsymbol{g}_0(z)$ occupy the first K columns of the above matrix: $\boldsymbol{G}_p^T(z) = \begin{bmatrix} \boldsymbol{g}_0(z) & \boldsymbol{\mathcal{G}}^T(z) \end{bmatrix}$. From the equation

$$\underbrace{\begin{bmatrix} \boldsymbol{h}_0^T(z) \\ \hat{\boldsymbol{\mathcal{H}}}(z) \end{bmatrix}}_{\hat{H}_p(z)} \underbrace{\begin{bmatrix} \boldsymbol{g}_0(z) & \boldsymbol{\mathcal{G}}^T(z) \end{bmatrix}}_{G_p^T(z)} = z^{-k}\begin{bmatrix} \boldsymbol{I}_K & \boldsymbol{0} \\ -\boldsymbol{U}^{-1}(z)\boldsymbol{E}(z) & \boldsymbol{U}^{-1}(z) \end{bmatrix}$$

it follows that $\boldsymbol{E}(z)$ should be given in terms of $\boldsymbol{U}(z)$ by[3]

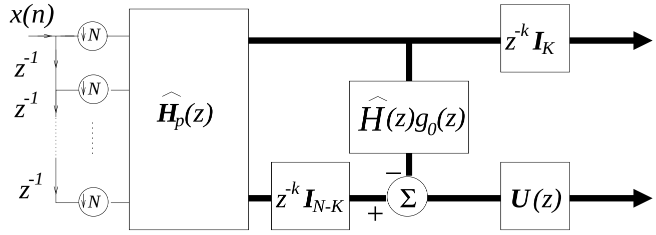
$$\boldsymbol{E}(z) = -z^k \boldsymbol{U}(z)\hat{\boldsymbol{\mathcal{H}}}(z)\boldsymbol{g}_0(z). \qquad (7)$$

The above considerations lead to the following:[4]

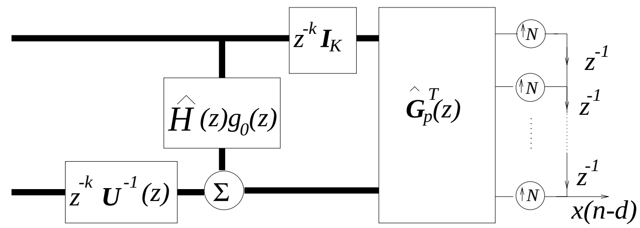
Theorem 2 *The set of N-band FIR PR ASS with the first K analysis and synthesis filters specified by the matrices $\boldsymbol{h}_0^T(z)$ and $\boldsymbol{g}_0^T(z)$ respectively, is generated by eqs. (5), (6) where $\boldsymbol{U}(z)$ is any $(N-K)\times(N-K)$ unimodular matrix and $\boldsymbol{E}(z)$ is given by (7).*

[3] In the special case of $K = N-1$ and $\boldsymbol{U}(z) = z^{-k}$, this simplifies to $\boldsymbol{E}(z) = -\hat{\boldsymbol{\mathcal{H}}}(z)\boldsymbol{g}_0(z)$ that is, the preselection of the first $N-1$ analysis and synthesis filters completely specifies the ASS, as expected.

[4] For a more insightful, but longer, proof the reader is referred to [8].



(a)



(b)

Figure 1: Ladder realization of the (a) analysis and (b) synthesis filter banks.

Since the matrix $\boldsymbol{E}(z)$ resulting from (7) may turn out to be noncausal, the insertion of an appropriate delay to the (analysis) filters may be required. A causal realization of the ASS is depicted in Fig. 1, where $d = 2kN + N - 1$.

3 DESIGN CONSIDERATIONS

We should emphasize that when $\boldsymbol{U}(z)$ ranges over the set of unimodular matrices of size $N-K$ (and determinant z^{-l} with $l \le k$), Fig. 1 yields a complete parameterization of the set of FIR PR ASS with the given filters at their first K bands, and provided that each of the blocks in those products is implemented as a cascade of ladder steps [9], it leads to a realization which preserves the PR property under both coefficient quantization and round-off noise. Note that the ladder factorization of the blocks $\hat{\boldsymbol{H}}_p(z)$ and $\hat{\boldsymbol{G}}_p^T(z)$ results naturally in the process of determining these matrices [7]. The resulting ladder-type structures provide a compact representation of the ASS and their property of PR preservation highly facilitates the subsequent optimization process. Nevertheless, a complete parameterization of the set of unimodular matrices of given determinant z^{-l} and order is still to be determined. It was shown in [13] that any such matrix can be expressed as the product of a paraunitary matrix with McMillan degree l and a *strictly* unimodular (i.e., of a constant determinant) matrix $\boldsymbol{\mathcal{U}}(z)$. The latter class of matrices (except for the 2×2 case) lacks a complete parameterization. To this end, one may employ induction on the matrix size and take advantage of Theorem 1 and the properties of the Euclidean algorithm [9], to arrive at a generic characterization (see also [7, Appendix A]). A related, incomplete, yet quite rich, factorization of the strictly unimodular matrices of size P and order n is derivable via matrix continued fraction expansions as in [6] and takes the form:

$$\boldsymbol{\mathcal{U}}(z) = \begin{bmatrix} -\boldsymbol{C}^{(2n+1)} & \boldsymbol{I}_p \\ \boldsymbol{I}_q & \boldsymbol{0} \end{bmatrix} \prod_{i=n}^{1} \boldsymbol{M}^{(i)}(z) \tag{8}$$

where

$$\boldsymbol{M}^{(i)}(z) \triangleq \begin{bmatrix} -\boldsymbol{C}^{(2i)}z^{-1} & \boldsymbol{I}_q \\ \boldsymbol{I}_p & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} -\boldsymbol{C}^{(2i-1)} & \boldsymbol{I}_p \\ \boldsymbol{I}_q & \boldsymbol{0} \end{bmatrix},$$

$\boldsymbol{C}^{(2i-1)}$ and $\boldsymbol{C}^{(2i)}$ are $p \times q$ and $q \times p$ matrices, respectively, and $p + q = P$.

Furthermore, the problem of designing the first K analysis and synthesis filters needs to be addressed. The fact that the matrices $\boldsymbol{h}_0^T(z)$ and $\boldsymbol{g}_0^T(z)$ must be irreducible [7] so that they can be completed up to unimodular matrices is not a matter of concern in practice, since this property can be shown to be generically satisfied [7]. It is not clear, however, how one can design these filters so that they also satisfy (3). The following gives us a clue to this question:[5]

Theorem 3 *The $K \times N$ matrices $\boldsymbol{h}_0^T(z)$ and $\boldsymbol{g}_0^T(z)$ satisfy (3) if and only if the $(N-1)$st polyphase component, $\boldsymbol{P}_{N-1}(z)$, of the $K \times K$ matrix polynomial $\boldsymbol{P}(z)$ defined by*

$$[P_{(i,j)}(z)] = \begin{bmatrix} H_0(z) \\ \vdots \\ H_{K-1}(z) \end{bmatrix} \begin{bmatrix} G_0(z) & \cdots & G_{K-1}(z) \end{bmatrix} \tag{9}$$

satisfies the equality:

$$\boldsymbol{P}_{N-1}(z) \triangleq [P_{(i,j),N-1}(z)] = z^{-k}\boldsymbol{I}_K. \tag{10}$$

Hence the problem of designing the first K bands can be viewed as one of designing a *multichannel Nth-band filter of rank 1.* In the more common case of $K = 1$, of interest in scalar wavelet design, the above reduces to the design of an Nth-band filter with desired properties and its subsequent factorization into the filters $H_0(z)$ and $G_0(z)$. In the context of wavelet design, $P(z)$ should be designed as a lowpass filter with a sufficiently great number of zeros at the frequencies $\frac{2\pi l}{N}$, $l = 1, 2, \ldots, N-1$, in order to obtain regular analysis and synthesis scaling filters. There are several approaches available for either LP (e.g., [4]) or non-LP (e.g., [15]) Nth-band filter design, able to incorporate regularity specifications. If n_c is such that $p(Nn + n_c) = \delta(n)$, it follows from Theorem 3 that $n_c = Nk + N - 1$ and the length of $p(n)$ has to be at least $N(k+1)$. For non-LP or nonpositive $P(z)$'s, care must be taken in choosing the factor filters so that they have sufficiently good frequency responses.[6]

[5] Proofs are omitted due to lack of space.

[6] Additional zeros on the unit circle may also be included in the design constraints. Such an approach has been seen to be potentionally more favourable to the design of smooth scaling functions than the increase of the regularity order [4].

The optimization criterion employed in our design experiments is the minimization of the stopband energies of the analysis and/or synthesis filters,[7] namely

$$\min_{U(z)} (\mathcal{E} = W_a\mathcal{E}_a + W_s\mathcal{E}_s) \tag{11}$$

where $\mathcal{E}_a = \sum_{i=K}^{N-1} \int_{\mathcal{S}_i} |H_i(e^{j\omega})|^2 d\omega$, $\mathcal{S}_i$ is the stopband of the ith filter, and similarly for $\mathcal{E}_s$. W_a, W_s are weighting constants reflecting our interest in analysis and synthesis filters' performance respectively.

4 THE LINEAR-PHASE CASE

Clearly, for a LP ASS, $\boldsymbol{P}(z)$ has to be designed so as to have also LP, and length $2(k+1)N - 1$. After splitting it into LP factors, the LP analysis (N, K)-problem can be solved via e.g., the approach of [2], which provides us moreover with a ladder realization of $\hat{\boldsymbol{H}}_p(z)$ and $\hat{\boldsymbol{G}}_p(z)$. The filter symmetries and lengths have to conform with the relevant constraints imposed by PR, as reported in [7, 2]. Moreover, it has to be taken into account that the synthesis filters of a PR ASS with LP analysis FB have also LP and the same symmetry types with their analysis counterparts [7].

Let $\hat{M}_i = \hat{m}_i N + I + 1$ be the lengths of the filters with polyphase matrix given by (4) and $M_i = m_i N + I + 1$ the lengths of the analysis filters to be designed ($0 \leq I \leq N-1$). Moreover, let $J_i \in \{-1, 1\}$ signify the type of symmetry of filter no. i. Then, it can be shown that:

Theorem 4 *The solutions to the LP analysis/synthesis (N, K)-problem are generated as in Fig. 1, where $\boldsymbol{U}(z)$ ranges over the set of $(N-K) \times (N-K)$ unimodular matrices satisfying*

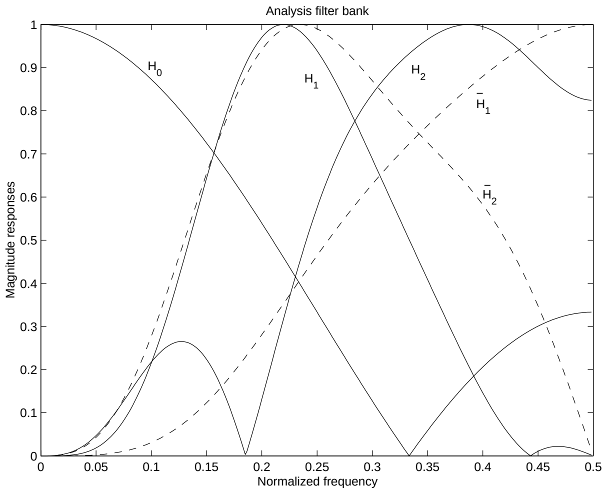
$$\boldsymbol{D}_{N-K}\boldsymbol{\Lambda}_{N-K}(z)\boldsymbol{U}(z) = z^{2k}\boldsymbol{U}(z^{-1})\boldsymbol{D}_{N-K}\hat{\boldsymbol{\Lambda}}_{N-K}(z) \tag{12}$$

where $\boldsymbol{D}_{N-K} = \mathrm{diag}(J_K, \ldots, J_{N-1})$, $\boldsymbol{\Lambda}_{N-K}(z) = \mathrm{diag}(z^{m_K}, \ldots, z^{m_{N-1}})$ and $\hat{\boldsymbol{\Lambda}}_{N-K}(z)$ is similarly defined. If $\det(\boldsymbol{U}(z)) = z^{-l}$, then $l \leq k$ and $2l = \sum_{i=K}^{N-1}(m_i - \hat{m}_i - 2k)$.

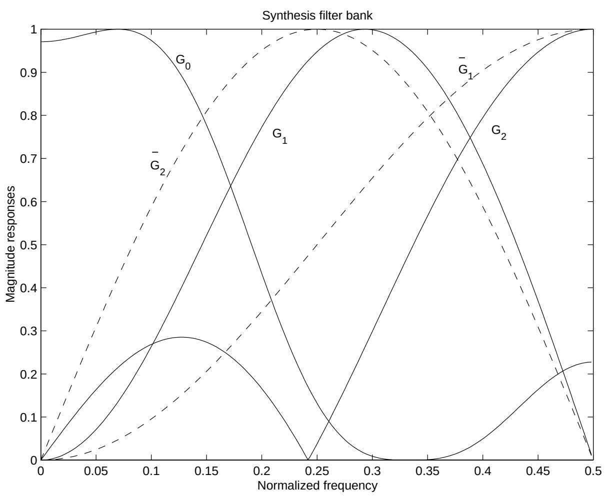
A parameterization of the above class of unimodular matrices (also referred to as *left-extension (l-ext)* matrices in [11]) can be derived as in [2, Appendix B]. The length of the filter $G_i(z)$ in the resulting ASS can be estimated as $L_i = 2\left(\sum_{j=0}^{N-1} m_j - 2l\right) - Nm_i + I + 1$.

Example: A simple, yet illustrative, example of the application of the above theory in the design of LP ASS is given by the 3-band ASS shown in Fig. 2, where $N = 3$ and $K = 1$. $P(z)$ was taken equal to the modulus squared of the 2-regular unitary filter of length 11 given in [12, Table I]. Its roots were re-distributed so as to yield LP factors $H_0(z)$ and $G_0(z)$ of regularity order 1 and 3 respectively. With the chosen orders m_i,

[7] It must be noted that, since the structure preserves the PR property, the passband errors need not be included in the optimization.



(a)



(b)

Figure 2: (a) Analysis and (b) synthesis filter banks of a 3-band linear phase PR ASS.

$\boldsymbol{U}(z)$ in (12) had to be of order 1 and its complete parameterization was used to optimize the frequency responses of filters 1 and 2, with $\mathcal{S}_1 = \left(0, \frac{\pi}{6}\right) \bigcup \left(\frac{5\pi}{6}, \pi\right)$, $\mathcal{S}_2 = \left(0, \frac{\pi}{2}\right)$ and $W_a = W_s = 1$. Fig. 2 depicts the magnitude responses of the filters before (dashed lines) and after (solid lines) the optimization.

5 CONCLUDING REMARKS

The theory presented is applicable to both single-channel and K-channel biorthogonal wavelet design [10]. Extensions to the multidimensional and IIR cases are also envisaged. Further study is required for the design of filters $\boldsymbol{P}(z)$ satisfying Theorem 3 with $K > 1$.

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