Properties of Some Blind Equalization Criteria in Noisy Multiuser Environments

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***Abstract*—Blind equalization in noisy multiuser channels has met with increasing attention with the advent of multiaccess digital communication systems. We examine blind equalizer performance in cases where perfect equalization proves unattainable due to noise and interference from concurrent users. In particular, we obtain a characterization of stationary points and extrema for a family of blind criteria in "undermodeled" cases, which assimilates the influence of differing source statistics and background noise correlation properties; relations to mean-square equalization measures are then obtained as a byprocuct. By re-examining a gradient search procedure, we obtain domains of attraction of each extremum in a special "sufficient order" setting. We also derive a global step-size bound for undermodeled cases, which ensures convergence of a gradient search procedure to an extremum of a blind cost function. We likewise confirm that the super-exponential algorithm results from an optimal choice of this step-size parameter.**

***Index Terms*—Blind equalization, minimum entropy methods, undermodeled equalizatiopn, multiaccess communications, super-exponential algorithm.**

I. Introduction

MANY techniques in blind equalization can be understood as minimum entropy methods first developed by Wiggins [1] and Donoho [2] and subsequently rediscovered [3] and refined [4] by Shalvi and Weinstein in a mono-source setting. Equivalences with the Godard [5] (or constant modulus [6]) criterion have since been placed in evidence [7], [8], as have relations with mutual information criteria and contrast functions [9]–[11]. This has motivated numerous contributions in a wide range of multisource signal separation and/or deconvolution settings [12]–[17].

A key result from [7] (mono-source case) and [13] (multi-source case) asserts that each extremum of a particular blind deconvolution criterion yields an ideal equalizer, i.e., giving a combined (channel-equalizer) impulse response having a sole nonzero term. The validity of this result, however, hinges strongly on the assumption that an arbitrary configuration of the combined (channel-equalizer) impulse response can be attained, including any ideal solution that would restore perfectly a transmitted sequence.

In practice, the presence of channel noise, as well as co-channel interference due to multiple users, will prohibit any equalizer setting from restoring perfectly a transmitted sequence of interest, and the restitution error tends to worsen with shorter equalizer lengths. The behavior of blind equalization algorithms in these "undermodeled" cases is not as clearly understood, as blind deconvolution criteria are usually nonconvex, exhibiting numerous local extrema.

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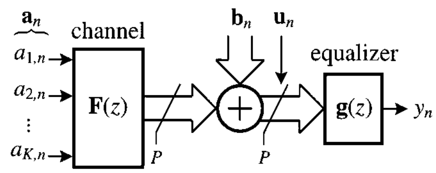

Fig. 1. Channel-equalizer cascade, including noise.

The intent of this work is to characterize the stationary points, local extrema, performance levels, and domains of convergence, for a family of blind deconvolution criteria, in a multisource noisy channel setting depicted in Fig. 1, using a finite (and generally insufficient) length equalizer. Section II reviews the problem structure while developing a model that assimilates the influence of spatially and temporally correlated channel noise. Section III derives a characterization of stationary points and local extrema for a family of blind criteria in the general undermodeled case. Section IV relates the performance level of a deconvolution criterion to traditional mean-square error measures. In Section V, we review a gradient descent procedure for which subsequent sections obtain, under appropriate conditions, domains of attraction of each convergent point and a step-size bound ensuring convergence to a local extremum for any initialization point. We verify, moreover, that the so-called super-exponential algorithm [4], [12] results from a certain "optimal" choice of the step size. Concluding remarks are synthesized in Section VIII.

II. Problem Structure

We consider a multichannel noisy deconvolution setting, which is depicted in Fig. 1, admitting a K-input–P-output discrete time baseband model of the form

$$(P\updownarrow) \quad \mathbf{u}_n = \mathbf{F}(z)\mathbf{a}_n + \mathbf{b}_n$$

where $\mathbf{u}_n$ is the observed vector process having P components, where P is the number of sensors times the oversampling factor. To ease notations, we assume real signals and channels. Complex extensions may be developed, provided signal circularity conditions are satisfied; otherwise, the real and imaginary parts of complex signals can be treated as separate real signals. Our model assumptions are as follows.

A1) The source signals $\{a_{i,n}\}$ that comprise the vector $\mathbf{a}_n$ are each independent, identically distributed (i.i.d.) random sequences and are mutually independent as well. We assume, with no loss of generality, that each is scaled to unit variance: $E[a_{i,n}^2] = 1$.

A2) The K-input–P-output transfer matrix $\mathbf{F}(z)$ is causal and stable

$$\mathbf{F}(z) = \sum_{k=0}^{\infty} \mathbf{F}_k z^{-k}, \quad \text{with } \sum_{k=0}^{\infty} \|\mathbf{F}_k\| < \infty.$$

Here, $\|\cdot\|$ denotes any matrix norm, and z^{-1} denotes the unit delay operator $z^{-1}\mathbf{a}_n = \mathbf{a}_{n-1}$. The product $\mathbf{F}(z)\mathbf{a}_n$ may then be interpreted as a convolution sum

$$\mathbf{F}(z)\mathbf{a}_n = \sum_{k=0}^{\infty} \mathbf{F}_k z^{-k}\mathbf{a}_n = \sum_{k=0}^{\infty} \mathbf{F}_k \mathbf{a}_{n-k}.$$

A3) The background noise vector $\mathbf{b}_n$ is Gaussian and independent of the source signals.

If the noise term is indeed Gaussian, then we may write an innovations model of the form

$$\mathbf{b}_n = \Phi(z)\alpha_n = \sum_{k=0}^{\infty} \Phi_k \alpha_{n-k}$$

where $\Phi(z)$ is the noise modeling filter, and the vector process $\{\alpha_n\}$ is normalized white noise

$$E\left[\alpha_n \alpha_m^T\right] = \begin{cases} \mathbf{I}, & n = m \\ \bigcirc, & n \neq m. \end{cases}$$

Since $\{\alpha_n\}$ is white and Gaussian, each sample is independent and identically distributed (i.i.d.). The special case where the background noise is spatially and temporally white is obtained with $\Phi(z) = \sigma\mathbf{I}$, where σ^2 is the noise variance. The general case of spatially and temporally correlated noise is accomodated by allowing more general choices for the noise modeling filter $\Phi(z)$. Note that if the noise is non-Gaussian, then an innovations model may still be developed using second-order statistics but will not yield a valid model for higher order statistics unless the noise process is linear. We therefore assume that the noise $\mathbf{b}_n$ is a linear process (which includes the Gaussian case) in the developments to follow.

Considering the noise term $\mathbf{b}_n$ thus as the output of a virtual channel $\Phi(z)$ driven by virtual sources α_n, we can combine the signal and noise terms into a common convolutional model

$$\begin{aligned} \mathbf{u}_n &= \mathbf{F}(z)\mathbf{a}_n + \Phi(z)\alpha_n \\ &= \underbrace{[\mathbf{F}(z) \quad \Phi(z)]}_{\mathcal{F}(z)} \begin{bmatrix} \mathbf{a}_n \\ \alpha_n \end{bmatrix} \\ &= \sum_{k=0}^{\infty} \underbrace{[\mathbf{F}_k \quad \Phi_k]}_{\mathcal{F}_k} \begin{bmatrix} \mathbf{a}_{n-k} \\ \alpha_{n-k} \end{bmatrix}. \end{aligned}$$

Some of the sources (namely, the $\{\alpha_{i,n}\}$) may be Gaussian, whereras the others (namely, the $\{a_{i,n}\}$) may be non-Gaussian.

The equalizer in Fig. 1 is a multi-input/single-output transversal filter

$$y_n = \sum_{k=0}^{M} \mathbf{g}_k \mathbf{u}_{n-k} = \sum_{k=0}^{M} \mathbf{g}_k z^{-k} \mathbf{u}_n = \mathbf{g}(z)\mathbf{u}_n.$$

Here, each impulse response term $\mathbf{g}_k$ is a row vector of P elements. We may write this in terms of the original source vector $\left[\begin{smallmatrix} \mathbf{a}_n \\ \alpha_n \end{smallmatrix}\right]$ as

$$\begin{aligned} y_n &= \mathbf{g}(z)\mathbf{u}_n \\ &= \mathbf{g}(z)\mathcal{F}(z) \begin{bmatrix} \mathbf{a}_n \\ \alpha_n \end{bmatrix} \\ &= \mathbf{s}(z) \begin{bmatrix} \mathbf{a}_n \\ \alpha_n \end{bmatrix} \\ &= \sum_{k=0}^{\infty} \mathbf{s}_k \begin{bmatrix} \mathbf{a}_{n-k} \\ \alpha_{n-k} \end{bmatrix}. \end{aligned}$$

Each term $\mathbf{s}_k$ is a row vector having as many entries as there are sources (both Gaussian and non-Gaussian). The impulse response sequence $\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, \ldots,$ is the *combined* (channel-equalizer) impulse response. It may be expressed as the convolution of the channel and equalizer impulse response sequences $\{\mathcal{F}_k\}_{k=0}^{\infty}$ and $\{\mathbf{g}_k\}_{k=0}^{M}$, respectively

$$\underbrace{\begin{bmatrix} \mathbf{s}_0^T \\ \mathbf{s}_1^T \\ \mathbf{s}_2^T \\ \vdots \\ \mathbf{s}_M^T \\ \mathbf{s}_{M+1}^T \\ \vdots \end{bmatrix}}_{\triangleq \mathbf{s}} = \underbrace{\begin{bmatrix} \mathcal{F}_0^T & \bigcirc & \bigcirc & \cdots & \bigcirc \\ \mathcal{F}_1^T & \mathcal{F}_0^T & \bigcirc & \cdots & \bigcirc \\ \mathcal{F}_0^T & \mathcal{F}_1^T & \mathcal{F}_0^T & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \bigcirc \\ \mathcal{F}_M^T & \mathcal{F}_{M-1}^T & \cdots & \mathcal{F}_1^T & \mathcal{F}_0^T \\ \mathcal{F}_{M+1}^T & \mathcal{F}_M^T & \cdots & \mathcal{F}_2^T & \mathcal{F}_1^T \\ \vdots & \ddots & \ddots & \ddots & \vdots \end{bmatrix}}_{\triangleq \mathcal{F}} \underbrace{\begin{bmatrix} \mathbf{g}_0^T \\ \mathbf{g}_1^T \\ \mathbf{g}_2^T \\ \vdots \\ \mathbf{g}_M^T \end{bmatrix}}_{\triangleq \mathbf{G}}.$$

An ideal equalizer setting would give a combined response $\mathbf{s}$ containing a sole nonzero term—positioned according to which source is to be deconvolved—if any such setting for $\mathbf{G}$ exists.

We observe that irrespective of how the equalizer coefficients $\{\mathbf{g}_k\}$ are chosen, the combined response vector $\mathbf{s}$ is restricted to the column space (in ℓ_2) of the channel convolution matrix $\mathcal{F}$; as in [7] and [18], we call this linear subspace the set of attainable combined responses, which is denoted $\mathcal{S}_A$

$$\mathcal{S}_A = \{\mathbf{s} : \mathbf{s} = \mathcal{F}\mathbf{G}, \text{ for some equalizer setting } \mathbf{G}\}.$$

The projection operator from ℓ_2 to $\mathcal{S}_A$ is denoted

$$\mathcal{P}_A = \mathcal{F}(\mathcal{F}^T\mathcal{F})^{\sharp}\mathcal{F}^T$$

where the superscript $\sharp$ denotes (pseudo-)inversion. A given $\mathbf{s}$ is then attainable $(\mathbf{s} \in \mathcal{S}_A)$ if and only if $\mathcal{P}_A\mathbf{s} = \mathbf{s}$. If $\mathcal{P}_A = \mathbf{I}$ (the identity), then an arbitrary configuration of the combined response vector $\mathbf{s}$ is attainable; this is called the *sufficient order* case [18]. If, on the other hand, $\mathcal{P}_A \neq \mathbf{I}$, then only a proper subset of ℓ_2 can be reached by varying the equalizer coefficients; this is called the *undermodeled* case.

Example 1: A sufficient order setting can result under special circumstances. For example, if the background noise vanishes, then the transfer matrix $\mathcal{F}(z)$ reduces to the "signal only" part $\mathbf{F}(z)$ of dimensions $P \times K$, where P is the product of the number of sensors times the oversampling factor, which is assumed larger than the number of sources K. If $\mathbf{F}(z)$ has polynomial entries (corresponding to a finite length impulse response for the channel) and has full rank for all z, then it may be understood as a submatrix of a $P \times P$ unimodular matrix [19]. The channel matrix $\mathbf{F}(z)$ then admits a left inverse that is also polynomial [19], corresponding to a perfect multioutput equalizer that separates and deconvolves all the sources. In this case, by choosing the equalizer length M sufficiently large (but dependent on the Kronecker indices associated with the channel, see [20]), the attainable part $\mathcal{S}_A$ of the combined response space coincides with the entire space, giving $\mathcal{P}_A = \mathbf{I}$. This is the setting of [13]–[15] in which multiple equalizers are run in parallel, possibly with cross coupling of the adaptation equations and/or source subtraction techniques to enforce decorrelation between the separate equalizer outputs and reduce the risk of two equalizers deconvolving the same source. The setting of [16] assumes a "doubly infinite" equalizer, i.e., having an infinite number of causal *and* anticausal impulse response terms. The combined response space is then composed of doubly infinite sequences for which the attainable part $\mathcal{S}_A$ will coincide with the entire space if $\mathbf{F}(z)$ has at least as many outputs as inputs and is devoid of Smith zeros on the unit circle. ◇

Example 2: A simpler setting involves an instantaneous mixture, which results when all channel impulse response terms except $\mathcal{F}_0$ vanish. Identifiability conditions typically involve the assumptions that the mixture matrix $\mathcal{F}_0$ has at least as many outputs as inputs and that at most one source signal is Gaussian (e.g., [9], [11]). Signal separation under these conditions may be accomplished using deflation approaches [21] or simultaneously adapting a multioutput "equalizer" based on contrast functions [11]. Performance characterizations treating the case of more inputs than outputs, interestingly, can be developed [22]–[25], particularly if separation rather than deconvolution is the goal [17]. ◇

The development to follow will, by contrast, allow nonzero noise, and invoke the (pessimistic) assumption that the noise innovation is a full-rank process, i.e., that the vector α_n has P components. In this case, the transfer matrix $\mathcal{F}(z)$ will have dimensions $P \times (K+P)$, giving fewer outputs than inputs, no matter how large P is chosen. The existence of a left inverse of $\mathcal{F}(z)$—which is critical to the validity of the deconvolution results of [13]–[16]—will not in general apply, even if the "noise-free" version of the channel were to fit the setting of Example 1.

III. Equalization Criterion

A commonly employed criterion for blind equalization [2], [3], [16], [18] involves normalized cumulants of the form

$$J_{2p} = \frac{\text{cum}_{2p}(y_n)}{[\text{cum}_2(y_n)]^p}, \qquad p = 2, 3, 4, \ldots$$

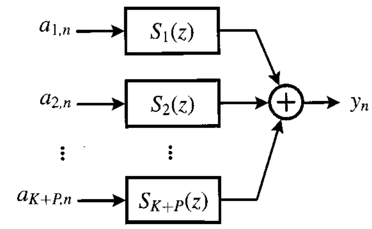

Fig. 2. Equivalent multi-input/single-output representation in terms of the combined transfer functions $S_i(z)$, one for each source.

in which $\text{cum}_{2p}(\cdot)$ denotes the cumulant of order $2p$ of the argument [26]; one seeks to maximize $|J_{2p}|$ by adjusting the equalizer coefficients. This family of objective functions may be derived from minimum entropy relations [2], from approximate mutual information relations (e.g., [10]), or from the theory of contrast functions [9], [11]. The order parameter p is user chosen; larger values of p imply higher order statistics, with a concommittant increase in estimation complexity, which favors choosing smaller values of p. The smallest choice $2p = 4$ is of particular interest, for which seeking the most negative value of J_4 may be shown [8] to be equivalent to minimizing the Godard [5] (or constant modulus [6]) cost function of the form $E[(1 - y_n^2)^2]$.

Now, suppose we have K sources $\{a_{i,n}\}$, $i = 1, 2, \ldots, K$, and let us rename the noise innovation sources $\{\alpha_{1,n}, \ldots, \alpha_{P,n}\}$ as $\{a_{K+1,n}, \ldots, a_{K+P,n}\}$. This allows us to write the equalizer output as

$$\begin{aligned} y_n &= \sum_{i=1}^{K+P} \sum_{k=0}^{\infty} s_{i,k} a_{i,n-k}, \qquad \begin{cases} i = \text{source index} \\ k = \text{time delay index} \end{cases} \\ &= \sum_{i=1}^{K+P} S_i(z) a_{i,n} \end{aligned}$$

in which $S_i(z) = \sum_{k=0}^{\infty} s_{i,k} z^k$ is the transfer function mapping the ith source a_i to the equalizer output y_n, as in Fig. 2.

As y_n is a weighted sum of independent random variables, it follows that [26]

$$\text{cum}_{2p}(y_n) = \sum_{i=1}^{K+P} \underbrace{\text{cum}_{2p}(a_{i,n})}_{\triangleq\, \gamma_i} \sum_{k=0}^{\infty} s_{i,k}^{2p}.$$

Note that if the noise innovation is Gaussian, then the corresponding cumulants vanish ($\gamma_{K+1} = \cdots = \gamma_{K+P} = 0$). Other linear noise processes are accommodated by allowing these cumulants to differ from zero. Nonlinear noise processes that cannot be written as a linearly filtered version of an i.i.d. sequence are not accommodated by this model.

Writing the equalization criterion in terms of the combined response $\mathbf{s}$ and the source cumulants $\{\gamma_i\}$ gives

$$J_{2p}(\mathbf{s}) = \frac{\text{cum}_{2p}(y_n)}{[\text{cum}_2(y_n)]^p} = \frac{\sum_{i=1}^{K+P} \gamma_i \sum_{k=0}^{\infty} s_{i,k}^{2p}}{\left(\sum_{i=1}^{K+P} \sum_{k=0}^{\infty} s_{i,k}^2\right)^p}. \tag{1}$$

If γ_- and γ_+ denote the most negative and most positive source cumulants, then $\gamma_- \le J_{2p} \le \gamma_+$. Moreover, J_{2p} is radially invariant: $J_{2p}(\beta \mathbf{s}) = J_{2p}(\mathbf{s})$ for all scalars $\beta \neq 0$. Without loss of generality, we may thus scale $\mathbf{s}$ to unit ℓ_2 norm: $\|\mathbf{s}\|_2 = 1$.

We may observe that the denominator of (1) is a convex function of the combined response $\mathbf{s}$. The numerator, on the other hand, will be convex (resp., concave) if the source cumulants $\{\gamma_i\}$ are all non-negative (resp., nonpositive). We call this the *sign definite cumulant* case. The *mixed cumulant* case (i.e., some cumulant values positive and others negative) will result in a numerator that is neither convex nor concave; this can lead to certain misconvergence difficulties as noted in previous contexts (e.g., [15], [11]) and later illustrated in Example 7. The results to follow apply to the mixed cumulant case, save for Section VII, which reverts to the sign definite cumulant case.

A. Stationary Points and Extrema Over $\mathcal{S}_A$

We now allow the combined response vector $\mathbf{s}$ to vary over the attainable subspace $\mathcal{S}_A$, and we seek the stationary points and extrema of the criterion J_{2p} from (1) in this subspace.

The directional derivative of the function J_{2p} at $\mathbf{s}$, with respect to a directional vector $\mathbf{r}$, is defined as (see [27, Sec. 23] and [28, ch. 7])

$$J'_{2p}(\mathbf{s};\mathbf{r}) \triangleq \lim_{t\to 0} \frac{J_{2p}(\mathbf{s}+t\mathbf{r}) - J_{2p}(\mathbf{s})}{t}$$

where t is a positive real scalar that tends to zero. Since the function J_{2p} from (1) is continuously differentiable in $\mathbf{s}$, one has the inner product form [27]

$$J'_{2p}(\mathbf{s};\mathbf{r}) = \langle \nabla J_{2p}(\mathbf{s}), \mathbf{r}\rangle$$

where $\nabla J_{2p}(\mathbf{s})$ is the gradient of J_{2p} at $\mathbf{s}$ (i.e., the vector whose entries are $\partial J_{2p}/\partial s_{i,k}$), and where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in ℓ_2: $\langle \mathbf{s},\mathbf{r}\rangle = \sum_{i,k} s_{i,k}\, r_{i,k}$.

Now, since the vector $\mathbf{s}+t\mathbf{r}$ spans $\mathcal{S}_A$ as $\mathbf{r}$ spans $\mathcal{S}_A$, a given vector $\mathbf{s}\in\mathcal{S}_A$ is a stationary point of J_{2p} over $\mathcal{S}_A$ if and only if the directional derivative of J_{2p} at $\mathbf{s}$ vanishes for all directional vectors $\mathbf{r}$ in $\mathcal{S}_A$

$$J'_{2p}(\mathbf{s};\mathbf{r}) = \langle \nabla J_{2p}(\mathbf{s}), \mathbf{r}\rangle = 0, \quad \text{for all } \mathbf{r}\in\mathcal{S}_A. \tag{2}$$

This says that $\nabla J_{2p}(\mathbf{s})$ is orthogonal to $\mathcal{S}_A$, i.e.,

$$\mathcal{P}_A \nabla J_{2p}(\mathbf{s}) = \mathbf{0}. \tag{3}$$

It now suffices to calculate the gradient using (1), but recalling our scaling assumption $\|\mathbf{s}\|_2 = 1$

$$[\nabla J_{2p}(\mathbf{s})]_{i,k} = \frac{\partial J_{2p}(\mathbf{s})}{\partial s_{i,k}} = 2p\left(\gamma_i s_{i,k}^{2p-1} - J_{2p}(\mathbf{s})\, s_{i,k}\right).$$

We may stack these scalars into a vector according to

$$\nabla J_{2p}(\mathbf{s}) = 2p\left(\begin{bmatrix} \gamma_1 s_{1,0}^{2p-1} \\ \vdots \\ \gamma_{K+P} s_{K+P,0}^{2p-1} \\ \gamma_1 s_{1,1}^{2p-1} \\ \vdots \\ \gamma_{K+P} s_{K+P,1}^{2p-1} \\ \vdots \end{bmatrix} - J_{2p}(\mathbf{s}) \begin{bmatrix} s_{1,0} \\ \vdots \\ s_{K+P,0} \\ s_{1,1} \\ \vdots \\ s_{K+P,1} \\ \vdots \end{bmatrix}\right)$$
$$= 2p(\mathcal{C}\mathbf{s}^{\odot(2p-1)} - J_{2p}(\mathbf{s})\,\mathbf{s})$$

in which we have the following.

- The vector $\mathbf{s}^{\odot(2p-1)}$ denotes the Hadamard power of order $2p-1$
$$\left[\mathbf{s}^{\odot(2p-1)}\right]_{i,k} = (s_{i,k})^{2p-1}.$$
- $\mathcal{C}$ is a diagonal matrix containing copies of the source cumulants of order $2p$
$$\mathcal{C} = \begin{bmatrix} \mathbf{C} & & \bigcirc \\ & \mathbf{C} & \\ \bigcirc & & \ddots \end{bmatrix}, \quad \text{with } \mathbf{C} = \begin{bmatrix} \gamma_1 & & \bigcirc \\ & \ddots & \\ \bigcirc & & \gamma_{K+P} \end{bmatrix}.$$

Now, since $\mathbf{s}$ is constrained to be attainable ($\mathbf{s}\in\mathcal{S}_A$), we have $\mathcal{P}_A\mathbf{s} = \mathbf{s}$, and the condition $\mathcal{P}_A \nabla J_{2p}(\mathbf{s}) = \mathbf{0}$ from (3) can be rephrased as the following theorem.

Theorem 1: A candidate $\mathbf{s}\in\mathcal{S}_A$ (scaled to unit ℓ_2 norm) is a stationary point of $J_{2p}(\mathbf{s})$ over $\mathcal{S}_A$ if and only if

$$\mathcal{P}_A\left(\mathcal{C}\mathbf{s}^{\odot(2p-1)}\right) = J_{2p}(\mathbf{s})\mathbf{s}.$$

Now, let "diag$(\mathbf{x})$" denote a diagonal matrix whose entries are the elements of a vector $\mathbf{x}$, and introduce the symmetric matrix

$$\mathcal{H}(\mathbf{s}) = \mathcal{P}_A\mathcal{C}\,\text{diag}\left(\mathbf{s}^{\odot(2p-2)}\right)\mathcal{P}_A. \tag{4}$$

If $\mathbf{s}$ is a stationary point of J_{2p} over $\mathcal{S}_A$ scaled to unit ℓ_2 norm, then we can observe that the scalar $J_{2p}(\mathbf{s})$ and the vector $\mathbf{s}$ form an eigenpair of $\mathcal{H}(\mathbf{s})$

$$\begin{aligned}\mathcal{H}(\mathbf{s})\mathbf{s} &= \mathcal{P}_A\mathcal{C}\,\text{diag}\left(\mathbf{s}^{\odot(2p-2)}\right)\mathcal{P}_A\mathbf{s} \quad \text{with } \mathcal{P}_A\mathbf{s}=\mathbf{s}\\ &= \mathcal{P}_A\mathcal{C}\mathbf{s}^{\odot(2p-1)} \quad \text{since diag}\left(\mathbf{s}^{\odot(2p-2)}\right)\mathbf{s} = \mathbf{s}^{\odot(2p-1)}\\ &= J_{2p}(\mathbf{s})\mathbf{s}, \quad \text{by Theorem 1.}\end{aligned} \tag{5}$$

Let λ_+ and λ_- denote, respectively, the most positive and most negative eigenvalues of $\mathcal{H}(\mathbf{s})$. The classification of $\mathbf{s}$ as a local extremum versus saddle point can be characterized as follows.

Theorem 2: Let $\mathbf{s}$ be a stationary point of J_{2p} over $\mathcal{S}_A$, with $\|\mathbf{s}\|_2 = 1$. If $J_{2p}(\mathbf{s}) > 0$ [resp., $J_{2p}(\mathbf{s}) < 0$], then, $\mathbf{s}$ lies at a local maximum (resp., local minimum) if and only if

$$J_{2p}(\mathbf{s}) = \lambda_+ \qquad (\text{resp., } J_{2p}(\mathbf{s}) = \lambda_-)$$

where the eigenvalue λ_+ (resp., λ_-) is simple and verifies the separation bound

$$\lambda_+ \geq (2p-1)\lambda_i \quad (\text{resp., } \lambda_- \leq (2p-1)\lambda_i)$$

for all other eigenvalues $\lambda_i \neq \lambda_+$ (resp., $\lambda_i \neq \lambda_-$) of $\mathcal{H}(\mathbf{s})$.

A proof is given in Appendix A.

Example 3: The solutions obeying Theorems 1 and 2 are explicit in the sufficient order case since $\mathcal{P}_A = \mathbf{I}$. The characterization of Theorem 1, for this case, simplifies to $\mathcal{C}\mathbf{s}^{\odot(2p-1)} = J_{2p}(\mathbf{s})\mathbf{s}$, which reads componentwise as

$$s_{i,k}(\gamma_i(s_{i,k})^{2p-2} - J_{2p}(\mathbf{s})) = 0, \quad \text{for all } i,k. \tag{6}$$

This says that all nonzero terms of $\mathbf{s}$, once scaled according to the source cumulants, share a common amplitude. The matrix $\mathcal{H}(\mathbf{s})$ from (4) assumes the diagonal form $\mathcal{C}\,\text{diag}(\mathbf{s}^{\odot(2p-2)})$, whose nonzero terms all equal $J_{2p}(\mathbf{s})$ in view of (6). This gives $J_{2p}(\mathbf{s}) = \lambda_+$ if $J_{2p}(\mathbf{s}) > 0$ or $J_{2p}(\mathbf{s}) = \lambda_-$ if $J_{2p}(\mathbf{s}) < 0$. If two or more terms of $\mathbf{s}$ are nonzero, the extremal eigenvalue of $\mathcal{H}(\mathbf{s})$ cannot be simple so that the stationary point cannot be a local extremum according to Theorem 2. Conversely, a sole nonzero term in $\mathbf{s}$, weighing a signal having a nonzero cumulant, leads to satisfaction of Theorem 2. This confirms that in the sufficient order case, a local extremum of $J_{2p}(\mathbf{s})$ is attained if and only if the combined response $\mathbf{s}$ has a sole nonzero term, allowing a source with nonzero cumulant to pass. ◇

IV. Relations to Mean-Square Performance Bounds

In undermodeled cases ($\mathcal{P}_A \neq \mathbf{I}$), perfect equalizers may not be attainable, and it is natural to examine whether traditional mean-square error criteria may be deduced from the value of J_{2p}. Here, we illustrate such relations for J_4.

Introduce a scaled criterion

$$\bar{J} = \begin{cases} J_4/\gamma_+, & \text{if } J_4 > 0 \\ J_4/\gamma_-, & \text{if } J_4 < 0 \end{cases}$$

so that $0 < \bar{J} \leq 1$ for any equalizer setting. Consider the polynomial $f(\zeta) = \zeta^2 - \zeta + (1/2)(1 - \bar{J})$, whose roots are

$$\zeta_- = \frac{1 - \sqrt{2\bar{J} - 1}}{2}, \quad \zeta_+ = \frac{1 + \sqrt{2\bar{J} - 1}}{2}. \tag{7}$$

We note that these roots become real when $\bar{J} \geq (1/2)$. The following result gives a lower bound on the quality of the dominant term of the combined response.

Theorem 3: Suppose $\bar{J} > (1/2)$ and that $\mathbf{s}$ is ℓ_2 normalized: $\|\mathbf{s}\|_2 = 1$. Then, precisely one term from the combined response (say, $s_{i,n}$) satisfies

$$s_{i,n}^2 \geq \zeta_+$$

and thus, the remaining terms satisfy

$$\sum_{(j,k)\neq(i,n)} s_{j,k}^2 \leq 1 - \zeta_+ = \zeta_-.$$

In particular, the interval (ζ_-, ζ_+) constitutes a "dead zone" that is devoid of the amplitude squared terms of a normalized combined response.

A proof is given in Appendix B.

The remainder of this section will examine the case $\bar{J} > (1/2)$, where we let $s_{i,n}$ denote the dominant term, which weighs the ith source with a delay of n samples. Let $\mathbf{e}_{i,n}$ be the unit vector with a "1" in position (i, n) and zeros elsewhere. The corresponding Wiener response, which is denoted $\mathbf{w}$, is the best least-squares approximation in $\mathcal{S}_A$ to $\mathbf{e}_{i,n}$, i.e., $\mathbf{w} = \mathcal{P}_A \mathbf{e}_{i,n}$. This is the (i, n) column (or row, by symmetry) of $\mathcal{P}_A$. For any attainable $\mathbf{s}$, we have $\mathcal{P}_A\mathbf{s} = \mathbf{s}$, whose (i, n) entry reads

$$s_{i,n} = \langle \mathbf{w}, \mathbf{s}\rangle, \quad \text{for all } \mathbf{s} \in \mathcal{S}_A.$$

This applies therefore to the choice $\mathbf{s} = \mathbf{w}$, giving $w_{i,n} = \|\mathbf{w}\|_2^2$. Since $\mathbf{e}_{i,n} - \mathbf{w}$ must be orthogonal to $\mathbf{w}$, the minimum mean-square error in deconvolving the ith source with a delay of n samples becomes

$$\text{MMSE} \triangleq \|\mathbf{e}_{i,n} - \mathbf{w}\|_2^2 = \|\mathbf{e}_{i,n}\|^2 - \|\mathbf{w}\|_2^2 = 1 - w_{i,n}. \tag{8}$$

With this, we can express some performance bounds in terms of $\bar{J}$.

Theorem 4: Suppose $\bar{J} \geq (1/2)$, and let $s_{i,n}$ denote the dominant term of $\mathbf{s}$.

1) The intersymbol interference measure is bounded as

$$\text{ISI} = \frac{\sum_{(j,k)\neq(i,n)} s_{j,k}^2}{s_{i,n}^2} \leq \frac{\zeta_-}{\zeta_+} = \frac{1 - \sqrt{2\bar{J} - 1}}{1 + \sqrt{2\bar{J} - 1}}.$$

2) If "MMSE" denotes the minimum mean-square error in deconvolving source i with delay n, then MMSE $\leq \zeta_-$ so that $\bar{J}$ is upper bounded as

$$\bar{J} \leq \frac{1}{2} + 2\left(\frac{1}{2} - \text{MMSE}\right)^2.$$

3) The subspace angle θ between $\mathbf{s}$ and the Wiener response $\mathbf{w}$ for source i at delay n obeys

$$\theta \leq \cos^{-1}\left(\sqrt{\frac{1 + \sqrt{2\bar{J} - 1}}{2(1 - \text{MMSE})}}\right).$$

We observe that the ISI, MMSE, and subspace angle bounds tend to zero as $\bar{J} \to 1$, as expected.

Proof: The first part is immediate from Theorem 3. For the second part, consider choosing a scale factor β to minimize $\|\mathbf{e} - \beta\mathbf{s}\|_2$; the optimal β is found as

$$\beta_{\text{opt}} = \langle \mathbf{e}, \mathbf{s}\rangle / \|\mathbf{s}\|_2^2 = s_{i,n}.$$

This renders $\mathbf{e} - \beta_{\text{opt}}\mathbf{s}$ orthogonal to $\mathbf{s}$ so that

$$\|\mathbf{e} - \beta_{\text{opt}}\mathbf{s}\|_2^2 = \|\mathbf{e}\|_2^2 - \|\beta_{\text{opt}}\mathbf{s}\|_2^2 = 1 - s_{i,n}^2 \leq \zeta_-.$$

This gives MMSE $\leq \zeta_-$ once it is recognized that $\mathbf{s}$ is not necessarily optimal in the mean-square error criterion. Rearranging the inequality MMSE $\leq \zeta_-$ to isolate $\bar{J}$ then completes the second part. Finally, by definition of subspace angle (e.g., [32])

$$\begin{aligned}\cos\theta &= \frac{|\langle \mathbf{w}, \mathbf{s}\rangle|}{\|\mathbf{w}\|_2 \cdot \|\mathbf{s}\|_2} \\ &= \frac{|s_{i,n}|}{\sqrt{1 - \text{MMSE}}\|\mathbf{s}\|_2} \geq \sqrt{\frac{\zeta_+}{1 - \text{MMSE}}}\end{aligned}$$

to give the final claim. ◇

V. Gradient Search Procedure

An extremum of $J_{2p}(\mathbf{s})$ can be approached using a gradient search procedure; we develop here the form such an algorithm may take and its relation to the super-exponential algorithm [4], [12], [29]. Convergence conditions of the algorithm will be developed in Section VI for the mixed cumulants but sufficient order case, and in Section VII for the undermodeled but sign-definite cumulant case.

Let $\mathbf{s}_{(0)}$ be an initial attainable setting in the combined response space, obtained from some equalizer initialization, and scaled to unit ℓ_2 norm. A gradient search procedure may be written in the combined response space as

$$\begin{aligned}\mathbf{v}_{(k+1)} &= \mathbf{s}_{(k)} \pm \frac{\mu_k}{2p}\mathcal{P}_A\left(\nabla J_{2p}(\mathbf{s}_{(k)})\right), \quad \mu_k > 0 \\ &= \mathbf{s}_{(k)} \pm \mu_k \mathcal{P}_A\left(\mathcal{C}\mathbf{s}_{(k)}^{\odot(2p-1)} - J_{2p}(\mathbf{s}_{(k)})\mathbf{s}_{(k)}\right) \\ \mathbf{s}_{(k+1)} &= \mathbf{v}_{(k+1)} / \|\mathbf{v}_{(k+1)}\|_2\end{aligned} \tag{9}$$

in which the ℓ_2 normalization of the final line is introduced because J_{2p} is radially invariant. The sign in front of μ_k is chosen according to whether the algorithm is to ascend $(\pm\mu_k \to +\mu_k)$

or descend ($\pm\mu_k \to -\mu_k$). We observe that with the particular step-size choice

$$\mu_k = \frac{1}{\left|J_{2p}\left(\mathbf{s}_{(k)}\right)\right|} \tag{10}$$

the gradient algorithm simplifies to

$$\mathbf{v}_{(k+1)} = \pm\mathcal{P}_A\mathcal{C}\mathbf{s}_{(k)}^{\odot(2p-1)}$$
$$\mathbf{s}_{(k+1)} = \mathbf{v}_{(k+1)} \Big/ \left\|\mathbf{v}_{(k+1)}\right\|_2$$

which may be recognized as the super-exponential algorithm (e.g., [4], [12], [29], [30]).

If we denote $\mathcal{U}_k = [\mathbf{u}_k^T, \ldots, \mathbf{u}_{k-M}^T]^T$, where M is the equalizer length, then the super-exponential algorithm admits a realizable form in the equalizer coefficient space as [4], [12], [30]

$$\mathbf{G}_{(k+1)} = \beta_k \left[\left(E\left[\mathcal{U}_k\mathcal{U}_k^T\right]\right)^{-1} \text{cum}[\mathcal{U}_k, \underbrace{y_k, \ldots, y_k}_{\substack{2p-1 \\ \text{terms}}}] \right]_{\mathbf{G}=\mathbf{G}_{(k)}}$$

where β_k is a constant that controls the radial factor of the equalizer vector. The general gradient algorithm (9) admits a realizable form as

$$\mathbf{G}_{(k+1)} = \beta_k \left[(1 \mp \mu_k J_{2p}(y_k))\mathbf{G}_{(k)} \pm \mu_k \left(E\left[\mathcal{U}_k\mathcal{U}_k^T\right]\right)^{-1} \text{cum}[\mathcal{U}_k, \underbrace{y_k, \ldots, y_k}_{\substack{2p-1 \\ \text{terms}}}] \right]_{\mathbf{G}=\mathbf{G}_{(k)}}.$$

In practice, of course, the various statistical quantities of these formulas must be replaced by empirical estimates; see [4], [12], and [30] for more on implementation aspects.

VI. Domains of Attraction in Sufficient Order Case

We deduce now the domains of attraction of each convergent point of the gradient search procedure in the sufficient-order case ($\mathcal{P}_A = \mathbf{I}$). A constraint commonly found in practical applications is that the nonzero source cumulants all have the same sign (e.g., [11], [13]–[15]). The results of this section, by contrast, apply to the general mixed cumulant case (i.e., having both positive and negative source cumulants). The fixed-point algorithm of [10] using a kurtosis contrast function coincides with the super-exponential algorithm adapted to the noiseless instantaneous mixture case; the convergence proof given in [10] likewise applies with mixed cumulants. The present development in addition specifies the domains of attraction of each convergent point as a function of the source cumulant values.

Analogously to [7], [13], and [18], we first consider a partition of the combined response space into convex cones, which is adapted here to the multisource case. Specifically, we will say that a given combined response $\mathbf{s}$ lies in the set $\mathcal{K}_{i,n}$ if $s_{i,n}$ is positive and dominant in the sense that

$$|\gamma_i|^{\frac{1}{2p-2}} s_{i,n} > \begin{cases} |\gamma_i|^{\frac{1}{2p-2}}|s_{i,k}|, & \text{for all } k \neq n \\ |\gamma_j|^{\frac{1}{2p-2}}|s_{j,k}|, & \text{for all } k \quad \text{and} \quad \text{all } j \neq i. \end{cases} \tag{11}$$

Observe that if $\mathbf{r}$ and $\mathbf{s}$ both lie in $\mathcal{K}_{i,n}$, then so does $\alpha\mathbf{r} + \beta\mathbf{s}$ for all positive constants α and β; the set $\mathcal{K}_{i,n}$ is, thus, a convex cone. Moreover, $\mathcal{K}_{i,n}$ is an open set; its closure is denoted $\bar{\mathcal{K}}_{i,n}$ and is obtained by replacing "$|\gamma_i|^{(1)/(2p-2)}s_{i,n} >$" in (11) with "$|\gamma_i|^{(1)/(2p-2)}s_{i,n} \geq$." We remark that if $s_{i,n}$ is negative and dominant, then $-\mathbf{s}$ will then lie in $\mathcal{K}_{i,n}$; since $J_{2p}(-\mathbf{s}) = J_{2p}(\mathbf{s})$, we may assume such "sign normalization" where convenient.

We also introduce a semi-norm as

$$\|\mathbf{s}\|_{\gamma} \triangleq \sup_{i,k}\left(|\gamma_i|^{\frac{1}{2p-2}}|s_{i,k}|\right)$$

which is just the ℓ_∞ norm of the weighted sequence $\{|\gamma_i|^{(1)/(2p-2)}s_{i,k}\}$. If, for the order $2p$ chosen, some cumulants γ_i vanish, then a nonzero $\mathbf{s}$ may have zero "norm" by this measure; thus, $\|\cdot\|_{\gamma}$ is a semi-norm. For this reason, we introduce a restricted domain $\mathcal{S}_{\gamma}$ as

$$\mathcal{S}_{\gamma} \triangleq \{\mathbf{s}: \text{ if } \gamma_i = 0 \text{ then } s_{i,k} = 0 \text{ for all } k\}.$$

The semi-norm then becomes a norm in the conventional sense in this restricted domain: If $\mathbf{s} \in \mathcal{S}_{\gamma}$, then $\|\mathbf{s}\|_{\gamma} = 0 \Leftrightarrow \mathbf{s} = \mathbf{0}$. Note that combined responses in this restricted domain eliminate Gaussian source contributions to the equalizer output.

Example 3 showed that for the sufficient-order case, if two or more terms are nonzero at a stationary point, their cumulant-weighted values $|\gamma_i|^{(1)/(2p-2)}|s_{i,k}|$ must all coincide. In view of (11), such a stationary point must lie on a cone boundary $\partial\mathcal{K}_{i,n}$ ($=\bar{\mathcal{K}}_{i,n} \ominus \mathcal{K}_{i,n}$). Since each open set $\mathcal{K}_{i,n}$ excludes its boundary, any stationary point within $\mathcal{K}_{i,n}$ must have a sole nonzero term, corresponding to an ideal combined response. If the corresponding cumulant value γ_i is nonzero, this ideal response is denoted $\mathbf{e}_{i,n}$

$$\mathbf{e}_{i,n} = \left[0 \cdots 0 \underbrace{|\gamma_i|^{\frac{-1}{2p-2}}}_{(i,n)} 0 \cdots \right]^T.$$

With $\gamma_i \neq 0$, we observe that $\mathbf{e}_{i,n} \in \mathcal{S}_{\gamma}$ and that $\mathbf{e}_{i,n}$ is scaled to have unit semi-norm: $\|\mathbf{e}_{i,n}\|_{\gamma} = 1$. For any other $\mathbf{s} \in \mathcal{K}_{i,n}$ scaled to unit semi-norm ($|\gamma_i|^{(1)/(2p-2)}s_{i,n} = \|\mathbf{s}\|_{\gamma} = 1$), we observe that $\mathbf{e}_{i,n}$ and $\mathbf{s}$ agree in position (i,n) so that their (semi-)distance must be less than one

$$\|\mathbf{e}_{i,n} - \mathbf{s}\|_{\gamma} = \sup_{(j,k)\neq(i,n)}\left(|\gamma_j|^{\frac{1}{2p-2}}|s_{j,k}|\right) < 1.$$

We may now show that the set $\mathcal{K}_{i,n}$ is a domain of attraction for the ideal response $\mathbf{e}_{i,n}$ in the sufficient-order case whenever $\gamma_i \neq 0$.

Let the subscript (k) denote an iteration index, and suppose that $\mathbf{s}_{(k)} \in \mathcal{K}_{i,n}$ and that $\gamma_i \neq 0$. Consider first the super-exponential algorithm [cf. (9) and (10)] specialized to the sufficient-order case ($\mathcal{P}_A = \mathbf{I}$)

$$\mathbf{s}_{(k+1)} = \mathrm{sgn}(\gamma_i)\mathcal{C}\mathbf{s}_{(k)}^{\odot(2p-1)}, \quad \mathrm{sgn}(\gamma_i) = \frac{\gamma_i}{|\gamma_i|}. \tag{12}$$

For convenience, we set $\mathbf{s} = \mathbf{s}_{(k)}$ and study the map

$$\mathbf{q} = \mathrm{sgn}(\gamma_i)\mathcal{C}\mathbf{s}^{\odot(2p-1)}.$$

Lemma 5: If $\mathbf{s} \in \mathcal{K}_{i,n}$ and $\mathbf{q} = \mathrm{sgn}(\gamma_i)\mathcal{C}\mathbf{s}^{\odot(2p-1)}$, then $\mathbf{q}$ remains in $\mathcal{K}_{i,n}$ and belongs to the restricted domain $\mathcal{S}_\sharp$. If $\|\mathbf{s}\|_\sharp = 1$, then $\|\mathbf{q}\|_\sharp = 1$, and moreover

$$\|\mathbf{e}_{i,n} - \mathbf{q}\|_\sharp = (\|\mathbf{e}_{i,n} - \mathbf{s}\|_\sharp)^{2p-1}.$$

A verification is given in Appendix C. We then obtain the following theorem.

Theorem 6: If $\gamma_i \neq 0$, then for any $\mathbf{s}_{(0)} \in \mathcal{K}_{i,n}$, the super-exponential algorithm (12) converges to the ideal response $\mathbf{e}_{i,n}$ in the sufficient-order case.

Indeed, the previous lemma shows that

$$\left\|\mathbf{e}_{i,n} - \mathbf{s}_{(k+1)}\right\|_\sharp = \left(\left\|\mathbf{e}_{i,n} - \mathbf{s}_{(k)}\right\|_\sharp\right)^{2p-1}.$$

We therefore obtain, by induction

$$\left\|\mathbf{e}_{i,n} - \mathbf{s}_{(k+1)}\right\|_\sharp = \left(\left\|\mathbf{e}_{i,n} - \mathbf{s}_{(0)}\right\|_\sharp\right)^{(2p-1)^{(k+1)}}.$$

This shows super-exponential convergence[1] toward $\mathbf{e}_{i,n}$ since, for any initial condition $\mathbf{s}_{(0)} \in \mathcal{K}_{i,n}$ with $\|\mathbf{s}_{(0)}\|_\sharp = 1$, we have $\|\mathbf{e}_{i,n} - \mathbf{s}_{(0)}\|_\sharp < 1$ and $\mathbf{e}_{i,n} - \mathbf{s}_{(k)} \in \mathcal{S}_\sharp$ for $k \geq 1$. ◇

Consider now the more general gradient search algorithm in the sufficient order case, viz.

$$\begin{aligned}\mathbf{v}_{(k+1)} &= [1 \mp \mu_k J_{2p}(k)]\mathbf{s}_{(k)} \pm \mu_k \mathcal{C}\mathbf{s}_{(k)}^{\odot(2p-1)} \\ \mathbf{s}_{(k+1)} &= \mathbf{v}_{(k+1)} \Big/ \left\|\mathbf{v}_{(k+1)}\right\|_\sharp \end{aligned} \tag{13}$$

in which "$\pm$"$=$ $\mathrm{sgn}(\gamma_i)$, and the second line uses $\sharp$-normalization for convenience. The following result shows conditions under which convergence applies even in the mixed cumulant case.

Theorem 7: Suppose that

i) $\mathbf{s}_{(0)} \in \mathcal{K}_{i,n}$ for some n;
ii) $\gamma_i \neq 0$;
iii) $\mathrm{sgn}\, J_{2p}(\mathbf{s}) = \mathrm{sgn}(\gamma_i)$ for all $\mathbf{s} \in \mathcal{K}_{i,n}$.

Then, $\mathbf{s}_{(k)} \to \mathbf{e}_{i,n}$ as $k \to \infty$ whenever $0 < \mathrm{sgn}(\gamma_i)\mu_k \leq 1/|J_{2p}(k)|$ for each k.

To verify, if $\mathrm{sgn}\, J_{2p}(\mathbf{s}_k) = \mathrm{sgn}\,\gamma_i$, then choosing $\pm\mu_k = \mathrm{sgn}(\gamma_i)|\mu_k|$ allows the update formula (13) to be rewritten as the convex sum

$$\lambda_k = \frac{|\mu_k|}{1 + |\mu_k|[1 - J_{2p}(k)]}.$$
$$\mathbf{s}_{(k+1)} = (1 - \lambda_k)\mathbf{s}_{(k)} + \lambda_k\left(\mathrm{sgn}(\gamma_i)\mathcal{C}\mathbf{s}_{(k)}^{\odot(2p-1)}\right). \tag{14}$$

[1]Using a logarithmic measure, as is often employed in numerical analysis, this corresponds to "super-linear" convergence, i.e., $\lim_{k\to\infty} \|\mathbf{e}_{i,n} - \mathbf{s}_{(k+1)}\|_\sharp / \|\mathbf{e}_{i,n} - \mathbf{s}_{(k)}\|_\sharp = 0$. For the practical case, $2p = 4$; this becomes cubic convergence.

Since $\mathbf{s}_{(k)}$ and $\mathrm{sgn}(\gamma_i)\mathcal{C}\mathbf{s}_{(k)}^{\odot(2p-1)}$ both lie in $\mathcal{K}_{i,n}$, so does $\mathbf{s}_{(k+1)}$ because $\mathcal{K}_{i,n}$ is a convex cone. As $\|\mathbf{s}_{(k+1)}\|_\sharp = 1$, its distance from $\mathbf{e}_{i,n}$ can be bounded using the triangle inequality of norms as

$$\begin{aligned}\left\|\mathbf{e}_{i,n} - \mathbf{s}_{(k+1)}\right\|_\sharp &= \Big\|(1-\lambda)\left(\mathbf{e}_{i,n} - \mathbf{s}_{(k)}\right) \\ &\quad + \lambda\left(\mathbf{e}_{i,n} - \mathrm{sgn}(\gamma_i)\mathcal{C}\mathbf{s}_{(k)}^{\odot(2p-1)}\right)\Big\|_\sharp \\ &\leq \left\|(1-\lambda_k)\left(\mathbf{e}_{i,n} - \mathbf{s}_{(k)}\right)\right\|_\sharp \\ &\quad + \lambda_k\left(\left\|\mathbf{e}_{i,n} - \mathbf{s}_{(k)}\right\|_\sharp\right)^{2p-1}\end{aligned}$$

in which equality may be shown to hold if all nonzero cumulants have the same sign. By induction on this inequality, we obtain

$$\left\|\mathbf{e}_{i,n} - \mathbf{s}_{(k+1)}\right\|_\sharp \leq \left\|\left(\mathbf{e}_{i,n} - \mathbf{s}_{(0)}\right)\left(\prod_{j=1}^{k+1}(1-\lambda_j)\right)\right\|_\sharp + \text{H.O.T.}$$

in which "H.O.T." collects all terms for which the exponent of $\|\mathbf{e}_{i,n} - \mathbf{s}_{(0)}\|_\sharp$ grows faster than linearly in the iteration index k. The first term on the right-hand side thus tends to zero the slowest. If $(1 - \lambda_j) \leq \beta < 1$ for all j, then the first term on the right-hand side is majorized by an exponentially decreasing sequence of the form $\beta^{k+1}\|\mathbf{e}_{i,n} - \mathbf{s}_{(0)}\|_\sharp$, and the convergence rate in this case is exponential, but no longer super-exponential. ◇

Example 4: If $\mathrm{sgn} J_{2p}(\mathbf{s}_{(k)}) \neq \mathrm{sgn}\gamma_i$, then choosing $\pm\mu_k = \mathrm{sgn}(\gamma_i)|\mu_k|$ can result in either λ_k or $1 - \lambda_k$ in (14) becoming negative so that $\mathbf{s}_{(k+1)}$ no longer needs to stay in $\mathcal{K}_{i,n}$. In the special case, however, where the source cumulants are sign definite (i.e., all non-negative or all nonpositive), the function $J_{2p}(\mathbf{s})$ is likewise sign definite, and condition iii) of Theorem 7 is satisfied automatically. Although in many applications the cumulant signs are known *a priori*, this indicates a potential weakness with respect to interfering sources having the "wrong" cumulant sign, forcing ascent where descent was intended, or vice-versa. ◇

VII. Step-Size Bound in Undermodeled Case

In the undermodeled case ($\mathcal{P}_A \neq \mathbf{I}$), the extrema of J_{2p} satisfying Theorems 1 and 2 are not readily obtained in closed form. This complicates efforts toward deducing domains of attraction of each extremum. We will nonetheless deduce a range for the step size μ_k that ensures convergence of the sequence $\{\mathbf{s}_{(k)}\}$ to an extremum of J_{2p}, irrespective of the initial condition $\mathbf{s}_{(0)}$, for the undermodeled case.

The stepsize bound is based on viewing the cost function $|J_{2p}(\mathbf{s})|$ as the ratio of two convex functions [29], which is applicable whenever all nonzero cumulants share the same sign. In the mixed cumulant case, by contrast, the relations with convexity are not, in general, valid [11]. Having cumulants sign definite ensures, in particular, that $J_{2p}(\mathbf{s})$ is also sign definite. In this case, the gradient procedure is to descend [$\pm\mu_k \to -\mu_k$ in (9)] if $J_{2p} \leq 0$ or ascend [$\pm\mu_k \to +\mu_k$ in (9)] if $J_{2p} \geq 0$. The following result extends that from [29] to the multisource setting; for notational convenience, we write $J_{2p}(k)$ for $J_{2p}(\mathbf{s}_{(k)})$.

Theorem 8: Suppose all cumulants are non-negative: $\gamma_i \geq 0$ (resp., all cumulants nonpositive: $\gamma_i \leq 0$). If $\mathbf{s}_{(k)}$ is not a stationary point of J_{2p}, the inequality $|J_{2p}(k+1)| > |J_{2p}(k)|$ holds in (9) whenever μ_k lies in the range

$$0 < \mu_k < \frac{2|J_{2p}(k)|}{2|J_{2p}(k)|^2 - \left\|\mathcal{P}_A \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)}\right\|_2^2}.$$

Example 5: Note that the upper bound on the step size μ_k varies with the position of the vector $\mathbf{s}_{(k)}$, as is typical when optimizing nonquadratic functions. A position-independent (but conservative) upper bound on μ_k can nonetheless be obtained.

We note first that for any attainable $\mathbf{s}_{(k)}$, the inequality $|J_{2p}(k)| \leq \|\mathcal{P}_A \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)}\|_2$ holds. Indeed, since $\mathbf{s}_{(k)}$ is scaled to unit ℓ_2 norm, the criterion $J_{2p}(k)$ may be written as

$$\begin{aligned} J_{2p}(k) &= \left\langle \mathbf{s}_{(k)}, \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)} \right\rangle \\ &= \left\langle \mathcal{P}_A \mathbf{s}_{(k)}, \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)} \right\rangle, \quad \text{since } \mathcal{P}_A \mathbf{s}_{(k)} = \mathbf{s}_{(k)} \\ &= \left\langle \mathbf{s}_{(k)}, \mathcal{P}_A \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)} \right\rangle \quad \text{by symmetry of } \mathcal{P}_A. \end{aligned}$$

The Cauchy–Schwarz inequality may then be invoked to give

$$\begin{aligned} |J_{2p}(k)| &= \left| \left\langle \mathbf{s}_{(k)}, \mathcal{P}_A \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)} \right\rangle \right| \\ &\leq \underbrace{\left\|\mathbf{s}_{(k)}\right\|_2}_{1} \cdot \left\| \mathcal{P}_A \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)} \right\|_2 \end{aligned}$$

with equality iff $\mathbf{s}_{(k)}$ and $\mathcal{P}_A \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)}$ are colinear (corresponding to a stationary point by Theorem 1). As such, the upper bound for μ_k can be lower bounded as

$$\frac{2|J_{2p}(k)|}{2|J_{2p}(k)|^2 - \left\|\mathcal{P}_A \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)}\right\|_2^2} \geq \frac{2}{|J_{2p}(k)|} \geq \frac{2}{\max_i |\gamma_i|}$$

since $|J_{2p}(k)| \leq \max_i |\gamma_i|$ at each iteration. (The left inequality becomes tight as a stationary point is approached). Any fixed step-size choice in the range $0 < \mu < 2/\max_i |\gamma_i|$—requiring knowledge only of an extremal cumulant value—will thus ensure convergence of $|J_{2p}|$ to a local maximum. ⋄

The proof of Theorem 8 treats the non-negative cumulant case, for which $0 \leq J_{2p} \leq \gamma_+$, since the nonpositive cumulant case follows by replacing $J_{2p}(\mathbf{s})$ by $-J_{2p}(\mathbf{s})$ and $+\mu_k$ by $-\mu_k$.

Let $\mathbf{x}$ be a free vector in ℓ_2, and introduce the scalar-valued function

$$N_{2p}(\mathbf{x}) = \sum_{i=1}^{K+P} \gamma_i \sum_{k=0}^{\infty} x_{i,k}^{2p} \geq 0$$

which assumes the same form as the numerator of J_{2p} from (1). Since the $2p$th root of $N_{2p}(\mathbf{x})$ is a weighted semi-norm, the function $N_{2p}(\mathbf{x})$ is convex, i.e., for all $\mathbf{x}_1$ and $\mathbf{x}_2$ in ℓ_2 and all $0 \leq \lambda \leq 1$

$$N_{2p}(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \leq \lambda N_{2p}(\mathbf{x}_1) + (1-\lambda) N_{2p}(\mathbf{x}_2).$$

As $N_{2p}(\mathbf{x})$ is continuous throughout ℓ_2, we can introduce its gradient $\nabla N_{2p}(\mathbf{x})$ as

$$\nabla N_{2p}(\mathbf{x}) = \begin{bmatrix} \partial N_{2p}(\mathbf{x})/\partial x_{1,0} \\ \vdots \\ \partial N_{2p}(\mathbf{x})/\partial x_{K+P,0} \\ \partial N_{2p}(\mathbf{x})/\partial x_{1,1} \\ \vdots \\ \partial N_{2p}(\mathbf{x})/\partial x_{K+P,1} \\ \vdots \end{bmatrix} = 2p \mathcal{C} \mathbf{x}^{\odot(2p-1)}.$$

Now, the (sub-)gradient inequality ([27, Sec. 23]) applicable to any convex function asserts that

$$N_{2p}(\mathbf{x} + \Delta\mathbf{x}) \geq N_{2p}(\mathbf{x}) + \langle \nabla N_{2p}(\mathbf{x}), \Delta\mathbf{x} \rangle$$

for all $\mathbf{x}$ and $\Delta\mathbf{x}$ in ℓ_2 (and not just for $\Delta\mathbf{x}$ chosen "small").

This inequality thus applies to the particular choices $\mathbf{x} = \mathbf{s}_{(k)}$ and $\mathbf{x} + \Delta\mathbf{x} = \mathbf{s}_{(k+1)}$, giving

$$N_{2p}\left(\mathbf{s}_{(k+1)}\right) - N_{2p}\left(\mathbf{s}_{(k)}\right) \geq 2p \left\langle \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)}, \mathbf{s}_{(k+1)} - \mathbf{s}_{(k)} \right\rangle.$$

Since both $\mathbf{s}_{(k)}$ and $\mathbf{s}_{(k+1)}$ have unit ℓ_2 norm, we can observe that N_{2p} and J_{2p} share a common evaluation

$$N_{2p}\left(\mathbf{s}_{(k)}\right) = \frac{N_{2p}\left(\mathbf{s}_{(k)}\right)}{\left\|\mathbf{s}_{(k)}\right\|_2^{2p}} = J_{2p}(k).$$

Similarly, $N_{2p}(\mathbf{s}_{(k+1)}) = J_{2p}(k+1)$ whenever $\|\mathbf{s}_{(k+1)}\|_2 = 1$. Our inequality involving N_{2p} can thus be rephrased in terms of J_{2p} as

$$J_{2p}(k+1) - J_{2p}(k) \geq 2p \left\langle \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)}, \mathbf{s}_{(k+1)} - \mathbf{s}_{(k)} \right\rangle \quad (15)$$

and it suffices to deduce which values of μ_k render the right-hand side positive. This takes the form

$$\left\langle \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)}, \underbrace{\frac{\mathbf{s}_{(k)} + \frac{\mu_k}{2p} \mathcal{P}_A \nabla J_{2p}\left(\mathbf{s}_{(k)}\right)}{\left\| \mathbf{s}_{(k)} + \frac{\mu_k}{2p} \mathcal{P}_A \nabla J_{2p}\left(\mathbf{s}_{(k)}\right) \right\|}}_{\mathbf{s}_{(k+1)}} \right\rangle > \underbrace{\left\langle \mathcal{C} \mathbf{s}_{(k)}^{\odot(2p-1)}, \mathbf{s}_{(k)} \right\rangle}_{J_{2p}(k)}.$$

Solving for μ_k compatible with this inequality gives the bound claimed in the theorem statement. ⋄

Example 6: The right-hand side of (15), when positive, represents a minimum increase in the function $|J_{2p}|$ at each iteration. The value of μ_k that maximizes this minimum increase is found by equating the derivative of the right-hand side of (15) with respect to μ_k to zero; the result gives a "min-max" optimal stepsize choice as $\mu_k^{\text{opt}} = 1/|J_{2p}(k)|$. This is the step-size value giving rise to the super-exponential algorithm; cf. (10). ⋄

Example 7: Although Theorem 8 ensures monotonic convergence for the sign-definite cumulant case, convergence difficul-

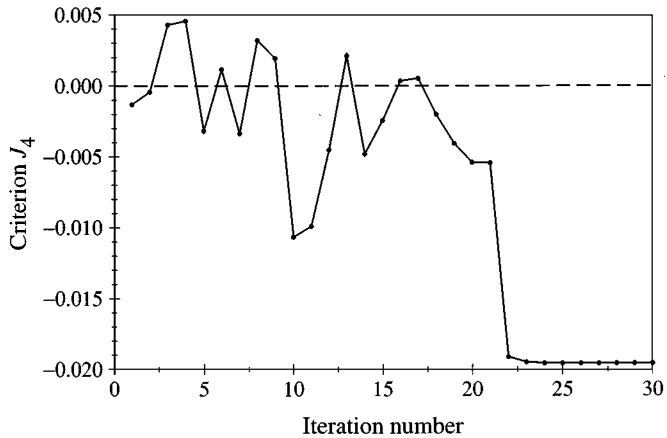

Fig. 3. Illustrating erratic behavior that may result in the mixed cumulant case.

ties may arise in the mixed cumulant case, as illustrated here. Consider a ten-input/three-output mixing matrix (transposed)

$$\mathcal{F}^t = \begin{bmatrix} -0.1715 & 0.0831 & 0.4182 \\ -0.1872 & -0.1616 & -0.1866 \\ -0.3150 & -0.4545 & -0.1110 \\ -0.4045 & 0.4289 & 0.0976 \\ 0.2018 & -0.3579 & 0.0885 \\ -0.0215 & 0.3922 & 0.0558 \\ 0.2476 & 0.3463 & 0.4318 \\ 0.3141 & 0.0860 & -0.3968 \\ 0.4923 & 0.2659 & -0.4020 \\ -0.4787 & 0.3106 & -0.5020 \end{bmatrix}$$

using ten sources having fourth-order cumulant values as

$$[\gamma_1, \ldots, \gamma_{10}] = [-0.38, -0.39, -0.23, -0.15, -0.10, 0.46, 0.12, 0.22, 0.04, -0.01]$$

and with an initial equalizer vector chosen as $\mathbf{g} = [-0.66, -0.70, 0.27]$. Fig. 3 shows the evolution of J_4 using the stepsize choice $\mu_k = 1/J_4(k)$. Although the algorithm does finally converge after about 20 iterations, its initial evolution is seen to be quite erratic.

VIII. Concluding Remarks

We have re-examined a class of minimum entropy deconvolution criteria in a multisource noisy channel setting. The results of this paper extend some earlier work [18], [29] developed in a monosource setting, obtaining a step-size range ensuring monotonic convergence of a gradient search procedure, and showing how the super-exponential algorithm results from an optimal choice of this step size. We have also obtained performance bounds in terms of mean-square error criteria. Our results apply to the general undermodeled case, which avoids assumptions that the source signals may be perfectly separated or deconvolved.

A potential weakness of the studied criteria is observed when the source cumulants are of mixed sign, i.e., some positive and others negative. The essential structure exploited in Section VII to obtain monotonic convergence involves writing the criterion $|J_{2p}|$ as the ratio of two convex functions, which is applicable when the source cumulants are sign definite. In the mixed cumulant case, this structure is lost, leading to potential difficulties induced by source cumulants having the "wrong" sign.

We may note, finally, that an adaptation of the "higher order power method" of [31] can, in principle, ensure monotonic convergence even with mixed cumulants, although the computational complexity of that algorithm is $2p$ times greater than that of the super-exponential algorithm; further developments in this direction will be reported in due course.

Appendix A
Proof of Theorem 2

A local extremum of $|J_{2p}(\mathbf{s})|$ corresponds to $\mathbf{s}$ attaining a local maximum of J_{2p} when this function is positive or attaining a local minimum of J_{2p} when this function is negative. We examine the former case, as the latter case follows by replacing $J_{2p}(\mathbf{s})$ with $-J_{2p}(\mathbf{s})$.

Introduce the second directional derivative of J_{2p} at $\mathbf{s}$, with repect to a directional vector $\mathbf{r}$, as

$$J''_{2p}(\mathbf{s};\mathbf{r}) = \left. \frac{d^2 J_{2p}(\mathbf{s}+t\mathbf{r})}{dt^2} \right|_{t=0}.$$

If $\mathbf{s}$ is a stationary point over $\mathcal{S}_A$, then it corresponds to a local maximum if and only if

$$J''_{2p}(\mathbf{s};\mathbf{r}) \le 0, \quad \text{for all } \mathbf{r} \in \mathcal{S}_A.$$

If we scale $\mathbf{s}$ to unit ℓ_2 norm, then a straightforward calculation shows that the second directional derivative takes the form

$$J''_{2p}(\mathbf{s};\mathbf{r}) = 2p \left(\underbrace{(2p-1)\left\langle \mathcal{C}\mathbf{s}^{\odot(2p-2)}, \mathbf{r}^{\odot 2} \right\rangle}_{(a)} - \underbrace{J_{2p}(\mathbf{s})\langle \mathbf{r},\mathbf{r}\rangle}_{(b)} - \underbrace{2(p-1)J_{2p}(\mathbf{s})\langle \mathbf{s},\mathbf{r}\rangle^2}_{(c)} \right).$$

Now, project $\mathbf{r}$ along $\mathbf{s}$

$$\mathbf{r} = \alpha\mathbf{s} + \mathbf{v}, \quad \text{with } \alpha = \langle \mathbf{r},\mathbf{s}\rangle \quad \text{and} \quad \langle \mathbf{v},\mathbf{s}\rangle = 0.$$

The three distinguished terms may then be handled in turn:

$$\begin{aligned} (c) &= 2(p-1)J_{2p}(\mathbf{s})\langle \mathbf{s},\mathbf{r}\rangle^2 = 2(p-1)J_{2p}(\mathbf{s})\alpha^2 \\ (b) &= J_{2p}(\mathbf{s})\langle \mathbf{r},\mathbf{r}\rangle = J_{2p}(\mathbf{s})(\alpha^2 + \langle \mathbf{v},\mathbf{v}\rangle) \\ (a) &= (2p-1)\langle \mathcal{C}\mathbf{s}^{\odot(2p-2)}, (\alpha^2\mathbf{s}^{\odot 2} + 2\alpha\mathbf{s}\odot\mathbf{v} + \mathbf{v}^{\odot 2})\rangle \\ &= (2p-1)\left(\alpha^2 J_{2p}(\mathbf{s}) + \left\langle \mathcal{C}\mathbf{s}^{\odot(2p-2)}, \mathbf{v}^{\odot 2}\right\rangle + 2\alpha \underbrace{\left\langle \mathcal{C}\mathbf{s}^{\odot(2p-1)}, \mathbf{v}\right\rangle}_{(d)} \right). \end{aligned}$$

Now, at any stationary point, Theorem 1 gives $\mathcal{C}\mathbf{s}^{\odot(2p-1)} = J_{2p}\mathbf{s} + \mathbf{b}$ with $\mathbf{b} \perp \mathcal{S}_A$. As such, term (d) appears as

$$(d) = \left\langle \mathcal{C}\mathbf{s}^{\odot(2p-1)}, \mathbf{v}\right\rangle = J_{2p}(\mathbf{s})\langle \mathbf{s},\mathbf{v}\rangle + \langle \mathbf{b},\mathbf{v}\rangle = 0$$

because both inner products vanish. Combining terms $(a), (b)$, and (c) now gives

$$J''_{2p}(\mathbf{s};\mathbf{r}) = 2p((2p-1)\left\langle \mathcal{C}\mathbf{s}^{\odot(2p-2)}, \mathbf{v}^{\odot 2}\right\rangle - J_{2p}(\mathbf{s})\langle \mathbf{v},\mathbf{v}\rangle)$$

in which we note that terms involving α^2 cancel. Therefore, a stationary point $\mathbf{s}$ will be a local maximum in $\mathcal{S}_A$ if and only if

$$(2p-1)\left\langle \mathcal{C}\mathbf{s}^{\odot(2p-2)}, \mathbf{v}^{\odot 2}\right\rangle \le J_{2p}(\mathbf{s}) \qquad (16)$$

for all unit-norm $\mathbf{v}$ that are in $\mathcal{S}_A$ and orthogonal to $\mathbf{s}$.

Consider now the symmetric matrix from (4)

$$\mathcal{H}(\mathbf{s}) = \mathcal{P}_A\mathcal{C}\,\mathrm{diag}(\mathbf{s}^{\odot 2})\mathcal{P}_A.$$

Since $\mathcal{P}_A\mathbf{v} = \mathbf{v}$ for any $\mathbf{v} \in \mathcal{S}_A$, the left-hand side of (16) appears as

$$(2p-1)\left\langle \mathcal{C}\mathbf{s}^{\odot(2p-2)}, \mathbf{v}^{\odot 2}\right\rangle = (2p-1)\mathbf{v}^t\mathcal{H}(\mathbf{s})\mathbf{v}.$$

We recall from (5) that at any stationary point $\mathbf{s}$, the vector $\mathbf{s}$ and scalar $J_{2p}(\mathbf{s})$ form an eigenpair of $\mathcal{H}(\mathbf{s})$. Since $\mathcal{H}(\mathbf{s})$ is symmetric, it admits a complete set of orthonormal eigenvectors; those eigenvectors corresponding to nonzero eigenvalues must lie in $\mathcal{S}_A$ since the range space of $\mathcal{H}(\mathbf{s})$ is contained in $\mathcal{S}_A$. Then, let $\mathbf{v}$ be any unit-norm eigenvector with a nonzero eigenvalue $\lambda_i \neq J_{2p}(\mathbf{s})$. Then, $\mathbf{v} \in \mathcal{S}_A$ and $\langle \mathbf{v},\mathbf{s}\rangle = 0$, and from (16), we will have

$$(2p-1)\lambda_i = (2p-1)\mathbf{v}^t\mathcal{H}(\mathbf{s})\mathbf{v} \le J_{2p}(\mathbf{s})$$

whenever $\mathbf{s}$ attains a local maximum. If $J_{2p}(\mathbf{s}) > 0$, then this holds if and only if i) $J_{2p}(\mathbf{s})$ is the most positive eigenvalue of $\mathcal{H}(\mathbf{s})$, and ii) the eigenvalue separation of Theorem 2 is verified. To show that $J_{2p}(\mathbf{s})$ is a simple eigenvalue of $\mathcal{H}(\mathbf{s})$, assume to the contrary that this eigenvalue has multiplicity two or greater. Since $\mathcal{H}(\mathbf{s})$ is symmetric, we can find another unit-norm eigenvector $\mathbf{v}$ with the same eigenvalue $J_{2p}(\mathbf{s})$ and othogonal to $\mathbf{s}$. If $J_{2p}(\mathbf{s}) > 0$, then

$$\begin{aligned}(2p-1)\left\langle \mathcal{C}\mathbf{s}^{\odot(2p-2)}, \mathbf{v}^{\odot 2}\right\rangle &= (2p-1)\mathbf{v}^t\mathcal{H}(\mathbf{s})\mathbf{v}\\ &= (2p-1)J_{2p}(\mathbf{s}) > J_{2p}(\mathbf{s})\end{aligned}$$

which contradicts (16). This completes the proof. ◇

APPENDIX B
PROOF OF THEOREM 3

Consider reindexing $\mathbf{s}^{\odot 2}$ as $t_j = s_{k,l}^2 \ge 0$, with $j = (K+P)l + k$, and suppose that $\mathbf{s}$ is scaled to unit ℓ_2 norm so that

$$\sum_{k,l} s_{k,l}^2 = \sum_j t_j = 1.$$

Assume $J_4 > 0$, as the case $J_4 < 0$ is treated analogously. Since $\gamma_+ = \max_i \gamma_i$, we have

$$\bar{J} = \frac{J_4}{\gamma_+} = \sum_{k,l}\frac{\gamma_k}{\gamma_+}s_{k,l}^4 \le \sum_{k,l}s_{k,l}^4 = \sum_j t_j^2 \triangleq \tau.$$

Now, each element t_n satisfies $t_n = 1 - \sum_{j\neq n} t_j$, and therefore

$$t_n^2 = 1 - 2\sum_{j\neq n} t_j + \left(\sum_{j\neq n} t_j\right)^2 \ge 1 - 2\sum_{j\neq n} t_j + \sum_{j\neq n} t_j^2.$$

Adding t_n^2 to either side gives

$$\begin{aligned}2t_n^2 &\ge 1 - 2\sum_{j\neq n} t_j + \underbrace{t_n^2 + \sum_{j\neq n} t_j^2}_{\tau}\\ &= 2\underbrace{\left(1 - \sum_{j\neq n} t_j\right)}_{t_n} + (\tau - 1) \ge 2t_n + (\bar{J} - 1)\end{aligned}$$

since $\bar{J} \le \tau$. A simple rearrangement then yields

$$t_n^2 - t_n + \frac{1-\bar{J}}{2} \ge 0, \quad \text{for all } n.$$

This shows that each term t_n must lie in an interval for which $f(\zeta) = \zeta^2 - \zeta + (1/2)(1-\bar{J})$ is non-negative. It is straightforward to check that with ζ_- and ζ_+ denoting the roots of $f(\zeta)$ [cf. (7)]

$$\text{when } \bar{J} > \frac{1}{2}, \quad f(\zeta) < 0, \quad \text{for } \zeta_- < \zeta < \zeta_+.$$

Therefore, each t_n must fulfill either $t_n \le \zeta_-$ or $t_n \ge \zeta_+$.

We note next that since $t_j \ge 0$ for all j

$$\bar{J} \le \sum_j t_j^2 \le \sup_j t_j \cdot \underbrace{\left(\sum_j t_j\right)}_{=1} = \sup_j t_j.$$

Our assumption $\bar{J} > 1/2$ then implies that at least one term from the sequence $\{t_j\}$ has amplitude greater than $1/2$. Since the value $1/2$ lies in the dead zone (ζ_-, ζ_+), at least one term must have amplitude no smaller than ζ_+.

Suppose now that two or more terms, say, t_l and t_m, have amplitude no smaller than ζ_+. Then

$$1 = \sum_k t_k \ge t_l + t_m \ge 2\zeta_+ > 1$$

which achieves a contradiction. We conclude that precisely one term from the sequence $\{t_k\}$ is no smaller than ζ_+. ◇

APPENDIX C
PROOF OF LEMMA 5

The verification is direct. If $\mathbf{s} \in \mathcal{K}_{i,n}$, then $s_{i,n}$ is positive and dominant, i.e.,

$$|\gamma_i|^{\frac{1}{2p-2}} s_{i,n} > |\gamma_j|^{\frac{1}{2p-2}}|s_{j,k}|, \quad \text{for all } (j,k) \neq (i,n).$$

Raising both sides to the power $2p-1$ preserves the inequality so that, for all $(j,k) \neq (i,n)$

$$\begin{aligned}\left(|\gamma_i|^{\frac{1}{2p-2}} s_{i,n}\right)^{2p-1} &> \left(|\gamma_j|^{\frac{1}{2p-2}}|s_{j,k}|\right)^{2p-1}\\ \Rightarrow |\gamma_i|^{\frac{2p-1}{2p-2}} s_{i,n}^{2p-1} &> |\gamma_j|^{\frac{2p-1}{2p-2}}|s_{j,k}|^{2p-1}\\ \Rightarrow |\gamma_i|^{\frac{1}{2p-2}} \underbrace{|\gamma_i| s_{i,n}^{2p-1}}_{q_{i,n}} &> |\gamma_j|^{\frac{1}{2p-2}} \underbrace{|\gamma_j s_{j,k}|^{2p-1}}_{|q_{j,k}|}.\end{aligned}$$

This shows that $\mathbf{q}$ remains in $\mathcal{K}_{i,n}$ and, moreover, that $\mathbf{q} \in \mathcal{S}_{\div}$ because if $\gamma_j = 0$, then $q_{j,k} = 0$.

If $\|\mathbf{s}\|_{\div} = 1$, then $s_{i,n} = |\gamma_i|^{(-1)/(2p-2)}$, in which case, an exercise shows that $q_{i,n} = |\gamma_i|^{(-1)/(2p-2)}$ as well, so that

$\|\mathbf{q}\|_{\natural} = 1$. Finally, since $\mathbf{e}_{i,n}$ and $\mathbf{q}$ agree in position (i,n), the distance $\|\mathbf{e}_{i,n} - \mathbf{q}\|_{\natural}$ can be evaluated as

$$\begin{aligned}\|\mathbf{e}_{i,n} - \mathbf{q}\|_{\natural} &= \sup_{(j,k)\neq(i,n)} \left(|\gamma_j|^{\frac{1}{2p-2}} |q_{j,k}|\right) \\ &= \sup_{(j,k)\neq(i,n)} \left(|\gamma_j|^{\frac{1}{2p-2}} |\gamma_j| \cdot |s_{j,k}|^{2p-1}\right) \\ &= \sup_{(j,k)\neq(i,n)} \left(|\gamma_j|^{\frac{2p-1}{2p-2}} |s_{j,k}|^{2p-1}\right) \\ &= \sup_{(j,k)\neq(i,n)} \left(|\gamma_j|^{\frac{1}{2p-2}} |s_{j,k}|\right)^{2p-1} \\ &= \left(\|\mathbf{e}_{i,n} - \mathbf{s}\|_{\natural}\right)^{2p-1}\end{aligned}$$

to complete the proof. $\diamond$

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