Past Input Reconstruction in Fast Least-Squares Algorithms

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***Abstract*—This paper solves the following problem: Given the computed variables in a fast least-squares prediction algorithm, determine all past input sequences that would have given rise to the variables in question. This problem is motivated by the backward consistency approach to numerical stability in this algorithm class; the set of reachable variables in exact arithmetic is known to furnish a stability domain. Our problem is equivalent to a first- and second-order interpolation problem introduced by Mullis and Roberts and studied by others. Our solution differs in two respects. First, relations to classical interpolation theory are brought out, which allows us to parametrize all solutions. By-products of our formulation are correct necessary and sufficient conditions for the problem to be solvable, in contrast to previous works, whose claimed sufficient conditions are shown to fall short. Second, our solution obtains any valid past input as the impulse response of an appropriately constrained orthogonal filter, whose rotation parameters derive in a direct manner from the computed variables in a fast least-squares prediction algorithm. Formulas showing explicitly the form of all valid past inputs should facilitate the study of what past input perturbation is necessary to account for accumulated arithmetic errors in this algorithm class. This, in turn, is expected to have an impact in studying accuracy aspects in fast least-squares algorithms.**

I. Introduction

FAST recursive least-squares filtering algorithms have been the subject of intense study in the past two decades in view of their widespread applicability to many practical filtering problems in modern signal processing. The vast body of literature devoted to this subject has been concerned with the "direct" problem: Given an input sequence, devise a computational algorithm that updates the relevant parameters of a fast least-squares prediction problem. This paper is devoted to the corresponding "inverse" problem: Given the values of the parameters of some fast least-squares prediction algorithm, deduce the set of all past input sequences that would have given rise to the values in question whenever such past inputs exist. This problem is intimately connected with backward consistency in least-squares algorithms, which plays an intriguing role in analyzing the error propagation properties of such algorithms [1]–[3]. This introductory section will review some basic concepts in order to motivate the study at hand.

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For the standard least squares filtering problem, one is given a sequence of input (row) vectors $\mathbf{u}(n)$, each comprising $M+1$ elements, plus a reference sequence $y(n)$. From these, introduce the data matrix and reference vector as

$$\mathbf{U}(n) = \begin{bmatrix} \mathbf{u}(n) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix}, \quad \mathbf{y}(n) = \begin{bmatrix} y(n) \\ \vdots \\ y(0) \end{bmatrix}.$$

A least-squares algorithm determines weights $\mathbf{h}(n) = [h_0(n), \ldots, h_M(n)]^t$ such that the resulting error vector $\mathbf{e}(n) = \mathbf{y}(n) - \mathbf{U}(n)\mathbf{h}(n)$ minimizes the cost function

$$\mathbf{e}^t(n)\Lambda(n)\mathbf{e}(n)$$

in which $\Lambda(n) = \text{diag}[1, \lambda, \ldots, \lambda^n]$ provides exponential weighting to past data, with $0 \ll \lambda \leq 1$.

Recursive least-squares algorithms compute the solution at time $n+1$ in terms of the variables from time n. In such algorithms, one typically finds time recursions involving the covariance matrix $\mathbf{P}(n) = \mathbf{U}^t(n)\Lambda(n)\mathbf{U}(n)$, or its inverse, or its Cholesky factor, or *its* inverse, etc., leading to order M^2 complexity at each time step.

Fast recursive least-squares algorithms apply when the vectors $\mathbf{u}(n)$ derive from a tapped delay line; the resulting $\mathbf{P}(n)$ inherits low displacement rank, e.g., [2], [5]. The matrix $\mathbf{P}(n)$ (or its inverse, or Cholesky factor, etc.) may then be parametrized with order M elements by exploiting forward and backward prediction; the time updates may then be reduced to order M computations. The prediction part of such fast algorithms takes the form of a time-recursive computation

$$\xi(n) = T[\xi(n-1), u_n] \tag{1}$$

in which

- $\xi(\cdot)$ state vector comprising the quantities that need to be written for storage at each iteration;
- u_n input (scalar) sample to the prediction part of the algorithm;
- $T[\cdot, \cdot]$ nonlinear map that implements the given fast least-squares subroutine at each iteration.

Although some of these fast algorithms suffer unstable error propagation, recent works [1]–[4] have shown that stable error propagation is enjoyed by any least-squares algorithm that is "backward consistent." In particular, let $\mathcal{S}_i$ denote the set of reachable states corresponding to the system (1);

this is the set of all state vector positions for $\xi(n)$ that are reachable in exact arithmetic as the past input u_n, u_{n-1}, and $u_{n-2}, \ldots$ is varied over all possibilities. Provided the state vector computed in finite precision—call it $\tilde{\xi}(n)$—never exits $\mathcal{S}_i$, the algorithm is termed backward consistent and enjoys stable error propagation. Indeed, if $\tilde{\xi}(n) \in \mathcal{S}_i$, then any errors in $\tilde{\xi}(n)$ are indistinguishable from errors on the past input, whose influence on the future evolution of system (1) will be exponentially decaying whenever $\lambda < 1$ and the future input $u_{n+1}, u_{n+2}, \ldots$ is persistently exciting. We refer the reader to [1]–[4] for more detail on this simple argument.

Deducing *necessary* conditions for a given state $\xi(\cdot)$ to be reachable in exact arithmetic involved exploiting known consistency conditions of least-squares algorithms [1], [2]. Showing these conditions to be sufficient as well required further work [4]. However, by definition of $\mathcal{S}_i$, if a state position $\xi(n)$ is reachable, then it must be possible to place in evidence some past input u_n, $u_{n-1}, u_{n-2}, \ldots$ that gives rise to this state. This paper will develop an analytic solution to the past input reconstruction problem, which consists of describing all past inputs that push the state to a given position whenever the position is indeed reachable. This provides a major supplement to the study of reachable states $\mathcal{S}_i$ initiated in [1].

This problem is equivalent to factoring the covariance matrix $\mathbf{P}(n)$ into a data matrix $\mathbf{U}(n)$ that displays the correct shift structure. Without this shift constraint, the factorization problem is elementary: A Cholesky factor will do, and any orthogonally transformed version of this Cholesky factor will also do. The Cholesky factor, though, will not in general display the correct shift structure; therefore, such an approach would not seem advantageous here. Conventional reachability concepts from system theory are difficult to apply directly in view of the nonlinear character of the map $T[\cdot,\cdot]$. An alternate attempt, which consists of trying to invert the system (1) in order to run it backward in time, is complicated by the problem of how to invert a many-to-one nonlinear map $T[\cdot,\cdot]$.

The solution developed here will instead show how all such past inputs may be obtained as the (anticausal) impulse response of an appropriately constrained orthogonal filter, whose rotation parameters derive in a fairly direct manner from the state variables $\xi(\cdot)$ of a fast least-squares algorithm. This ensures, in particular, a stable past input reconstruction scheme.

The paper is organized as follows. Section II reviews the structure of the factorization problem at hand, to point out its relation to [6]–[8], as well as the shortcoming of the claimed sufficient conditions for solvability of those works. Section III shows how our factorization problem may be solved by way of a related interpolation problem of the Schur type. Section IV shows how the key parameters in the past input reconstruction problem derive from the state variables of fast least squares algorithms, whereas Section V provides perspectives for future work.

II. Problem Structure

Fast least-squares algorithms apply when the input vector derives from a scalar sequence passed through a delay line $\mathbf{u}(n) = [u_n \; u_{n-1} \; \cdots \; u_{n-M}]$. If $u_n = 0$ for $n < 0$, the matrix $\mathbf{U}(n)$ then assumes a "prewindowed" Hankel structure

$$\mathbf{U}(n) = \begin{bmatrix} u_n & u_{n-1} & \cdots & u_{n-M} \\ u_{n-1} & u_{n-2} & \cdots & u_{n-M-1} \\ u_{n-2} & u_{n-3} & \cdots & u_{n-M-2} \\ \vdots & \iddots & \iddots & \vdots \\ u_M & \cdots & u_1 & u_0 \\ \vdots & \iddots & u_0 & 0 \\ u_1 & \iddots & \iddots & \vdots \\ u_0 & 0 & \cdots & 0 \end{bmatrix}. \tag{2}$$

Its exponentially weighted gramian is $\mathbf{P}(n) = \mathbf{U}^t(n) \times \Lambda(n)\mathbf{U}(n)$. Consider first $\lambda = 1$. Introduce the correlation lags

$$r_k = \sum_{i=0}^{n-k} u_i u_{i+k}, \quad k = 0, 1, \ldots, M \tag{3}$$

and rename the most recent input samples as

$$q_1 = u_n, \quad q_2 = u_{n-1}, \quad \ldots \quad q_M = u_{n-M+1}. \tag{4}$$

Then, for any n, we have the structured matrix

$$\begin{aligned}\mathbf{P}(n) &= \mathbf{U}^t(n)\mathbf{U}(n) \\ &= \begin{bmatrix} r_0 & r_1 & \cdots & r_M \\ r_1 & r_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_1 \\ r_M & \cdots & r_1 & r_0 \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \iddots & 0 & q_1 \\ 0 & \iddots & \iddots & \vdots \\ 0 & q_1 & \cdots & q_M \end{bmatrix}^2 \end{aligned} \tag{5}$$

as a direct exercise will verify. This consists of a symmetric Toeplitz matrix minus the square of a triangular Hankel matrix.

For $\lambda < 1$, let $\mathbf{L} = \operatorname{diag}[1, \lambda^{1/2}, \ldots, \lambda^{M/2}]$. Because $\mathbf{U}(n)$ is a Hankel matrix, we can write

$$\Lambda^{1/2}(n)\mathbf{U}(n) = \bar{\mathbf{U}}(n)\mathbf{L}^{-1}$$

in which $\bar{\mathbf{U}}(n)$ is a Hankel matrix akin to (2) but built from the sequence

$$\bar{u}_{n-i} = \lambda^{i/2} u_{n-i}, \quad i = 0, 1, \ldots, n. \tag{6}$$

As such, $\mathbf{LP}(n)\mathbf{L} = \bar{\mathbf{U}}^t(n)\bar{\mathbf{U}}(n)$ is the gramian of a Hankel matrix and may thus be written as a structured matrix like (5). Factoring $\mathbf{P}(n)$ when $\lambda < 1$ may thus be reduced to an equivalent problem as though $\lambda = 1$ had been used, and the past input had been exponentially weighted, as in (6), so that we may consider the case $\lambda = 1$ with no loss of generality.

A. Common Parametrizations

The parameters $\{r_k\}$ and $\{q_k\}$ are not those commonly encountered in recursive least-squares algorithms, and accordingly, we review their connections with more commonly used parameters.

1) Fast Transversal Filters: The fast transversal equations (with their many variants) are well defined only when $\mathbf{P}(n)$ is positive definite and, hence, invertible. The inverse $\mathbf{P}^{-1}$ (time index n suppressed) is known to have low displacement rank according to [2]

$$\begin{bmatrix}\mathbf{P}^{-1} & \mathbf{0}\\ \mathbf{0}^t & 0\end{bmatrix} - \begin{bmatrix}0 & \mathbf{0}^t\\ \mathbf{0} & \mathbf{P}^{-1}\end{bmatrix} = \begin{bmatrix}\mathbf{A}_M\\ 0\end{bmatrix}[\mathbf{A}_M^t \quad 0] + \begin{bmatrix}\mathbf{C}_M\\ 0\end{bmatrix}[\mathbf{C}_M^t \quad 0] - \begin{bmatrix}0\\ \mathbf{B}_M\end{bmatrix}[0 \quad \mathbf{B}_M^t] \tag{7}$$

in which

$$\mathbf{A}_M^t = \frac{[1 \ \ -a_1 \ \ \cdots \ \ -a_M]}{\sqrt{\alpha_M}}$$

contains the forward prediction coefficients $\{a_k\}$, the forward prediction error energy α_M

$$\mathbf{B}_M^t = \frac{[-b_M \ \ \cdots \ \ -b_1 \ \ 1]}{\sqrt{\beta_M}}$$

contains the backward prediction coefficients $\{b_k\}$ and the backward prediction error energy β_M, and

$$\mathbf{C}_M^t = \frac{[0 \ \ c_1 \ \ \cdots \ \ c_M]}{\sqrt{\gamma_M}}$$

contains the Kalman gain vector coefficients $\{c_k\}$ and the conversion factor γ_M. The fast transversal family of algorithms are based on performing time updates not on the matrix $\mathbf{P}^{-1}(n)$ but on the corresponding generator vectors $\mathbf{A}_M(n)$, $\mathbf{B}_M(n)$, and $\mathbf{C}_M(n)$; these variables in turn yield the elements of the state vector $\xi(n)$ in such algorithms.

2) Order Recursive Algorithms: Suppose $\mathbf{P}(n)$ is truncated to its $(k+1)\times(k+1)$ principal submatrix; the resulting matrix, once inverted and displaced akin to (7), yields generator vectors $\mathbf{A}_k(n)$, $\mathbf{B}_k(n)$, and $\mathbf{C}_k(n)$, each of $k+1$ elements. For any order k, introduce the generator functions

$$\begin{aligned} A_k(z) &= [1 \quad z \quad \cdots \quad z^k]\mathbf{A}_k(n)\\ B_k(z) &= [1 \quad z \quad \cdots \quad z^k]\mathbf{B}_k(n)\\ C_k(z) &= [1 \quad z \quad \cdots \quad z^k]\mathbf{C}_k(n). \end{aligned}$$

These polynomials are then related by the order recursion [5]

$$\begin{bmatrix}A_{k+1}(z)\\ C_{k+1}(z)\\ B_{k+1}(z)\end{bmatrix} = \begin{bmatrix}\frac{1}{\cos\phi_k} & & \frac{\sin\phi_k}{\cos\phi_k}\\ & 1 & \\ \frac{\sin\phi_k}{\cos\phi_k} & & \frac{1}{\cos\phi_k}\end{bmatrix} \times \begin{bmatrix}1 & & \\ & \frac{\sin\theta_k}{\cos\theta_k} & \frac{1}{\cos\theta_k}\\ & \frac{1}{\cos\theta_k} & \frac{\sin\theta_k}{\cos\theta_k}\end{bmatrix}\begin{bmatrix}A_k(z)\\ C_k(z)\\ zB_k(z)\end{bmatrix} \tag{8}$$

in which $\sin\phi_k$ is the correlation coefficient (or parcor) between the normalized forward and backward prediction error residuals of degree k, and $\sin\theta_k$ is a "doubly normalized" backward prediction error of degree k, meaning the angle normalized error divided by the square root of the corresponding backward prediction error energy. These, of course, correspond to the lattice or fast QR state variables to be updated in time, although many variants exist (e.g., [3], [5], [9]–[11]).

B. Shift Invariance

Since the structure exposed in (5) holds for any n, the number of rows of any Hankel factor $\mathbf{U}(n)$ obtained from $\mathbf{P}(n)$ is indeterminate. To better appreciate this indeterminacy, we review the shift invariance of this problem. To this end, suppose the values assumed by the parameters $\{r_k\}_{k=0}^M$ and $\{q_k\}_{k=1}^M$ are reachable at time n, i.e., there exists some input sequence $\{u_i\}_{i=0}^n$ fulfilling (3) and (4). Then, these same values are reachable at time $n+1$, using the sequence $\{\tilde{u}_i\}_{i=0}^{n+1}$ defined by a simple shift operation

$$\{\tilde{u}_{n+1}, \tilde{u}_n, \ldots, \tilde{u}_1, \tilde{u}_0\} = \{u_n, u_{n-1}, \ldots, u_0, 0\}.$$

Conversely, any values for $\{r_k\}_{k=0}^M$ and $\{q_k\}_{k=1}^M$ that are reachable at time $n+1$ are also reachable at time n, provided we push the starting time from $i=0$ back to $i=-1$. The set of *asymptotically reachable* values may be understood as those reachable by fixing the starting time at $i=0$ and letting the final time extend to $n=+\infty$ or, equivalently, by fixing the final time to $n=-1$ and letting the starting time extend back to $i=-\infty$. The latter convention affords two advantages. First, the z-transform of any valid past input sequence takes the form

$$U(z) = \sum_{i=1}^{\infty} u_{-i}\, z^i. \tag{9}$$

Whenever a past input exists, it must be square summable [provided $r_0 < \infty$ in (3)] so that $U(z)$ will be analytic in $|z| < 1$. Second, the set of all reachable values $\{r_k\}$ and $\{q_k\}$ will yield precisely the set of all valid initial conditions at time $n=-1$ for the fast least-squares algorithm to proceed correctly from time $n=0$ onward.[1]

Setting thus the final time freely to $n=-1$, the past input reconstruction problem becomes the following.

Problem 1: Given the structured matrix

$$\mathbf{P}(-1) = \begin{bmatrix} r_0 & r_1 & \cdots & r_M\\ r_1 & r_0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & r_1\\ r_M & \cdots & r_1 & r_0\end{bmatrix} - \begin{bmatrix}0 & \cdots & 0 & 0\\ \vdots & \cdot^{\cdot^{\cdot}} & 0 & q_1\\ 0 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots\\ 0 & q_1 & \cdots & q_M\end{bmatrix}^2 \tag{10}$$

find all anticausal functions $U(z)$ as in (9) that satisfy the interpolation constraints

$$u_{-k} = q_k, \quad k = 1, 2, \ldots, M; \tag{11}$$

$$\sum_{i=1}^{\infty} u_{-i}u_{-i-k} = r_k, \quad k = 0, 1, \ldots, M. \tag{12}$$

[1] Stated otherwise, the "biased solution" resulting from any such valid initial condition will yield the "exact solution" of an augmented problem that takes into account the past input that deposited the given initial condition at time $n = -1$.

This is precisely the problem introduced and studied by Mullis and Roberts [6]: Design a discrete-time linear system [$U(z)$ in our context] with constraints on the leading terms of its impulse response and autocorrelation sequences. See also Inouye [7] and King *et al.* [8] for multivariable extensions.

These works claim that a solution exists if and only if the values of $\{r_k\}$ and $\{q_k\}$ give rise to a $\mathbf{P}(-1)$, which is nonnegative definite (i.e., having no negative eigenvalues). Although not recognized in [6] and [8], a solution need not exist when $\mathbf{P}(-1)$ is positive semi-definite and singular. As an example, choose $q_1 = \cdots q_M = 0$, and then choose the terms r_k to yield a positive semi-definite (and singular) Toeplitz matrix; no ℓ_2 sequence $\{u_{-k}\}$ can then exist fulfilling (12). A less obvious example is obtained by choosing $r_0 = 1$, $r_1 = \frac{1}{2}$, $r_2 = \frac{3}{4}$ and $q_1 = -\frac{1}{2}$, $q_2 = \frac{1}{4}$, leading to the structured matrix

$$\mathbf{P}(-1) = \begin{bmatrix} 1 & 0.5 & 0.75 \\ 0.5 & 1 & 0.5 \\ 0.75 & 0.5 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -0.5 \\ 0 & -0.5 & 0.25 \end{bmatrix}^2. \quad (13)$$

This matrix is positive semi-definite of rank two, and the "Toeplitz part" built from r_0, r_1, and r_2 is positive definite. The results of [6] and [8] claim that $U(z)$ exists and yields a rational function of degree two [$= \text{rank}\mathbf{P}(-1)$]. Either method, though, leads to

$$\begin{aligned} U(z) &= \frac{1}{2}\frac{-z + z^2}{1 - 0.5z - 0.5z^2} \\ &= \frac{1}{2}\frac{-z(1-z)}{(1+0.5z)(1-z)} = \frac{-0.5z}{1+0.5z}. \end{aligned}$$

We will further expose this example in Section III-B and show that although the first-order constraint (11) is satisfied, the second-order constraint (12) is not, and that the difficulty stems from a pole-zero cancellation on the unit circle (at $z = 1$ in this example). Neither of these examples is excluded by the claimed data consistency contraints of [6] or [8], which reveals a shortcoming of the claimed sufficient conditions for solvability.

The solution we shall develop is inspired from [12] and [13]. In this light, Problem 1 is set in the Hardy space $\mathcal{H}_2$ in that we seek a function $U(z)$ that is analytic in $|z| < 1$ with bounded L_2 norm on the unit circle (by way of $r_0 = \|U(z)\|_2^2$). The approaches of [12] or [13] allow one to convert this to an $\mathcal{H}_\infty$ problem in that the interpolation constraints involve a function analytic in $|z| < 1$ and bounded by one there, which bounds the L_∞ norm on $|z| = 1$ by way of the maximum modulus principle. The analytic theory of such $\mathcal{H}_\infty$ interpolation problems is quite mature [16], [17], and the solution may then be transformed back to solve the original $\mathcal{H}_2$ problem [12], [13]. The following section shows how this strategy may be adapted to the present problem.

III. A Related Interpolation Problem

Let $\mathcal{Z}$ be the shift matrix with ones on the subdiagonal and zeros elsewhere. The matrix $\mathbf{P}(-1)$ has low displacement rank, and its displacement residue $\mathbf{P}(-1) - \mathcal{Z}\mathbf{P}(-1)\mathcal{Z}^t$ is readily seen to be

$$\begin{aligned} &\mathbf{P}(-1) - \mathcal{Z}\mathbf{P}(-1)\mathcal{Z}^t \\ &= \begin{bmatrix} r_0 & r_1 & \cdots & r_M \\ r_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ r_M & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ q_1 \\ \vdots \\ q_M \end{bmatrix} [\cdot]^t \\ &= \begin{bmatrix} \sqrt{r_0} \\ r_1/\sqrt{r_0} \\ \vdots \\ r_M/\sqrt{r_0} \end{bmatrix} [\cdot]^t - \begin{bmatrix} 0 \\ r_1/\sqrt{r_0} \\ \vdots \\ r_M/\sqrt{r_0} \end{bmatrix} [\cdot]^t - \begin{bmatrix} 0 \\ q_1 \\ \vdots \\ q_M \end{bmatrix} [\cdot]^t \end{aligned} \quad (14)$$

where "$[\cdot]$" means "repeat the previous vector."

Now, behind many a displacement structure lurks an interpolation problem [14], [15]; that corresponding to (14) may be introduced as follows.

Let $\mathbf{S}(z)$ be a 2×1 vector-valued Schur function, meaning that $\mathbf{S}(z)$ is analytic in $|z| < 1$ and contractive there, i.e., $\|\mathbf{S}(z)\| < 1$ for all $|z| < 1$ (using the Euclidean norm $\|\cdot\|$). Let us set

$$a(z) \triangleq \sqrt{r_0} + \frac{r_1}{\sqrt{r_0}} z + \cdots + \frac{r_M}{\sqrt{r_0}} z^M \quad (15)$$

$$\begin{bmatrix} c(z) \\ b(z) \end{bmatrix} = \mathbf{S}(z) a(z) \quad (16)$$

in which $\mathbf{S}(z)$ is to be designed as follows.

Problem 2: Given the values of the parameters $\{r_k\}_{k=0}^M$ and $\{q_k\}_{k=1}^M$, find a Schur function $\mathbf{S}(z)$ such that the resulting $b(z)$ and $c(z)$ assume the forms

$$c(z) = 0 + q_1 z + q_2 z^2 + \cdots + q_M z^M + O_1(z^{M+1}) \quad (17)$$

$$b(z) = 0 + \frac{r_1}{\sqrt{r_0}} z + \cdots + \frac{r_M}{\sqrt{r_0}} z^M + O_2(z^{M+1}) \quad (18)$$

where $O(z^{M+1})$ denotes a function analytic in $|z| < 1$, which vanishes $M+1$ times at $z = 0$.

By a classic result [16], [18], a solution to this problem exists if and only if a certain Pick matrix is nonnegative definite. The Pick matrix in question is known [16] to solve a Lyapunov–Stein equation relating to the interpolation data; this equation is simply (14). Thus, Problem 2 admits a solution $\mathbf{S}(z)$ if and only if $\mathbf{P}(-1) \geq \bigcirc$.

Now, since $b(z)$ and $c(z)$ both vanish at $z = 0$, whereas $a(z)$ does not, we see that any solution $\mathbf{S}(z)$ must vanish at $z = 0$, which allows us to write $\mathbf{S}(z) = \left[\begin{smallmatrix} zS_1(z) \\ zS_2(z) \end{smallmatrix}\right]$. Whenever a solution exists, then at least one *lossless* solution exists (e.g., [16]), where lossless refers to a Schur function that has unit norm along the unit circle $z = e^{j\omega}$

$$|S_1(e^{j\omega})|^2 + |S_2(e^{j\omega})|^2 = 1, \quad \text{for all } \omega. \quad (19)$$

The following proposition summarizes all solutions to Problem 1.

Proposition 3: Let $\mathbf{S}(z)$ be a lossless solution to Problem 2. If the resulting $zS_2(z)$ obeys the constraint

$$1 - zS_2(z) \neq 0, \quad \text{for all } |z| = 1 \quad (20)$$

then the function

$$U(z) = \sqrt{r_0} \frac{zS_1(z)}{1 - zS_2(z)} \quad (21)$$

is a solution to Problem 1. Moreover, all solutions to Problem 1 may be generated in this way.

The case in which $1 - z\,S_2(z) = 0$ for some $z = e^{j\omega}$ induces a type of singularity in the construct, which will be examined in Section III-B. The constraint (20) did not enter into the formulations of [6]–[8], the absence of which explains the shortcoming of the claimed sufficient conditions in those works.

The proof of this proposition requires familiarity with positive real functions, for which a review of certain properties, adapted from [19]–[22], may prove helpful to some readers. Let $\mathcal{R}(z)$ be analytic in $|z| < 1$ and written as the power series

$$\mathcal{R}(z) = \frac{\mathcal{R}_0}{2} + \sum_{k=1}^{\infty} \mathcal{R}_k z^k, \quad |z| < 1.$$

This function is positive real, provided its real part is positive at all points inside the unit circle.

- By using the Cayley transform, any positive real function can be expressed as

$$\mathcal{R}(z) = \frac{\mathcal{R}_0}{2}\frac{1 + zS_2(z)}{1 - zS_2(z)}$$

  where $S_2(z)$ is a Schur function (sometimes called a scattering function [21]).
- If $S_2(z)$ is a rational allpass function, then $\mathcal{R}(z)$ will be called a *reactance function*. Such a function has all poles and zeros interlaced on the unit circle [22].
- The spectrum associated with a positive real function is twice the real part of $\mathcal{R}(z)$ along the unit circle and is defined from the radial limit

$$\mathcal{R}(e^{j\omega}) + \mathcal{R}(e^{-j\omega}) \triangleq \lim_{\rho \to 1} (\mathcal{R}(\rho e^{j\omega}) + \mathcal{R}(\rho e^{-j\omega})) \geq 0.$$

  If $\mathcal{R}(z)$ is a reactance function, then this limit vanishes for almost all ω, except when ω is an angular location of a pole of $\mathcal{R}(z)$; for that case, the limit tends to $+\infty$. The spectrum of a reactance function thus consists of Dirac delta functions on the unit circle located at the poles of $\mathcal{R}(z)$. Such a spectrum is not factorable.
- In case $S_2(z)$ is not an allpass function but the condition $1 - e^{j\omega_k} S_2(e^{j\omega_k}) = 0$ is nonetheless produced a finite number of times (N, say), then the positive real function $\mathcal{R}(z)$ may be split as

$$\mathcal{R}(z) = \mathcal{R}_s(z) + \mathcal{R}_a(z)$$

  where $\mathcal{R}_s(z)$ is a reactance function containing the N unit-circle poles of $\mathcal{R}(z)$, and $\mathcal{R}_a(z)$ is a positive real function free from poles on the unit circle [21], [22]. The reactance function $\mathcal{R}_s(z)$ will be called the *singular part* and $\mathcal{R}_a(z)$ the *absolutely continuous part* of the positive real function $\mathcal{R}(z)$.[2] If $\mathcal{R}_s(z)$ vanishes, the spectrum is absolutely continuous, positive-valued for almost all ω and factorable.

[2]More properly, the spectra corresponding to $\mathcal{R}_s(z)$ and $\mathcal{R}_a(z)$ constitute the singular and absolutely continuous parts of the spectrum corresponding to $\mathcal{R}(z)$.

A. Proof of Proposition 3

We shall assume in this section that the constraint (20) is satisfied and provide a verification that $U(z)$ in (21) indeed solves Problem 1. Beginning with a lossless solution $\mathbf{S}(z) = \left[\begin{smallmatrix} zS_1(z) \\ zS_2(z)\end{smallmatrix}\right]$ to Problem 2, we construct the positive real function

$$\begin{aligned}\mathcal{R}(z) &= \frac{r_0}{2}\frac{1 + zS_2(z)}{1 - zS_2(z)} \\ &= \frac{\mathcal{R}_0}{2} + \mathcal{R}_1 z + \mathcal{R}_2 z^2 + \cdots \quad |z| < 1. \end{aligned} \tag{22}$$

By constraint (20), $\mathcal{R}(z)$ has no unit-circle poles; the spectrum corresponding to $\mathcal{R}(z)$ is absolutely continuous and thus factorable.

Let us first show that

$$\mathcal{R}_k = r_k \quad k = 0, 1, \ldots, M \tag{23}$$

i.e., that this positive real function interpolates the given correlation data. To this end, we observe that by construction of $S_2(z)$ in Problem 2, we have $b(z) = z\,S_2(z) a(z)$. Consider

$$\begin{aligned} a(z) + b(z) &= [1 + zS_2(z)]a(z) \\ a(z) - b(z) &= [1 - zS_2(z)]a(z). \end{aligned} \tag{24}$$

Upon eliminating $a(z)$ from the right-hand side of either equation, we obtain

$$\frac{r_0}{2}(a(z) + b(z)) = \underbrace{\frac{r_0}{2}\frac{1 + zS_2(z)}{1 - zS_2(z)}}_{\mathcal{R}(z)}(a(z) - b(z)) \tag{25}$$

where

$$\begin{aligned} a(z) - b(z) &= \sqrt{r_0} + 0z + \cdots + 0z^M - O_2(z^{M+1}) \\ \frac{r_0}{2}(a(z) + b(z)) &= \sqrt{r_0}\Big(\frac{r_0}{2} + r_1 z + \cdots \\ &\qquad + r_M z^M + O_2(z^{M+1})\Big). \end{aligned}$$

We may write (25) in the time domain as a convolution, whose first $M{+}1$ terms become

$$\sqrt{r_0}\begin{bmatrix} r_0/2 \\ r_1 \\ r_2 \\ \vdots \\ r_M \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\mathcal{R}_0 & 0 & 0 & \cdots & 0 \\ \mathcal{R}_1 & \frac{1}{2}\mathcal{R}_0 & 0 & \cdots & 0 \\ \mathcal{R}_2 & \mathcal{R}_1 & \frac{1}{2}\mathcal{R}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \mathcal{R}_M & \cdots & \mathcal{R}_2 & \mathcal{R}_1 & \frac{1}{2}\mathcal{R}_0 \end{bmatrix} \begin{bmatrix} \sqrt{r_0} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This gives (23).

In the same way, we may insert (24) into (16) and, recalling the form of $U(z)$ from (21), write

$$\sqrt{r_0}c(z) = \underbrace{\sqrt{r_0}\frac{zS_1(z)}{1 - zS_2(z)}}_{U(z)}(a(z) - b(z)).$$

In the time domain, the first $M{+}1$ terms become

$$\sqrt{r_0}\begin{bmatrix} 0 \\ q_1 \\ q_2 \\ \vdots \\ q_M \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ u_{-1} & 0 & 0 & \cdots & 0 \\ u_{-2} & u_{-1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ u_{-M} & \cdots & u_{-2} & u_{-1} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{r_0} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This gives $u_{-k} = q_k$ for $k = 1, 2, \ldots, M$.

The verification will be complete on showing that $U(z)$ is a spectral factor associated with $\mathcal{R}(z)$, i.e., $U(z^{-1})U(z) = \mathcal{R}(z^{-1}) + \mathcal{R}(z)$, since this will imply that the correlation lags from $U(z)$ will be $\mathcal{R}_k$ $(= r_k$ for $k = 0, 1, \ldots, M)$. Indeed, by a direct calculation, we have

$$\begin{aligned}\mathcal{R}(z^{-1}) &+ \mathcal{R}(z)\\ &= \frac{r_0}{2}\frac{1+z^{-1}S_2(z^{-1})}{1-z^{-1}S_2(z^{-1})} + \frac{r_0}{2}\frac{1+zS_2(z)}{1-zS_2(z)}\\ &= r_0\frac{1-S_2(z^{-1})\,S_2(z)}{[1-z^{-1}S_2(z^{-1})][1-zS_2(z)]}\\ &= r_0\frac{S_1(z^{-1})S_1(z)}{[1-z^{-1}S_2(z^{-1})][1-zS_2(z)]} = U(z^{-1})U(z)\end{aligned}$$

in which the second-to-last line is obtained from the previous one thanks to (19).

B. On the Supplementary Condition

Suppose now that condition (20) is violated, i.e., there exists some $z = e^{j\omega_0}$ such that $1 - e^{j\omega_0}S_2(e^{j\omega_0}) = 0$. The function $\mathcal{R}(z)$ from (22) then has a pole on the unit circle at $z = e^{j\omega_0}$. We have, in particular, that $|S_2(e^{j\omega_0})| = 1$, which implies $S_1(e^{j\omega_0}) = 0$ by way of (19).

The formula for $U(z)$ from (21) then reveals a pole-zero cancellation on the unit circle at $z = e^{j\omega_0}$. Repeating the above analysis will then show that $\mathcal{R}(z)$ still interpolates the correlation data $r_0, \ldots, r_M$ and that $U(z)$ still generates the terms $q_1, \ldots, q_M$. However, the equality $\mathcal{R}(z^{-1}) + \mathcal{R}(z) = U(z^{-1})U(z)$ can no longer hold since $\mathcal{R}(z)$ contains a unit-circle pole, whereas $U(z)$ does not.

To illustrate, consider the lossless function of degree two

$$\mathbf{S}(z) = \begin{bmatrix} zS_1(z) \\ zS_2(z)\end{bmatrix} = \begin{bmatrix} z(z-1)/2 \\ z(z+1)/2\end{bmatrix}.$$

We observe that $zS_2(z)|_{z=1} = 1$ so that condition (20) is violated. Choosing $r_0 = 1$, the positive real function $\mathcal{R}(z)$ becomes

$$\begin{aligned}\mathcal{R}(z) &= \frac{r_0}{2}\frac{1+zS_2(z)}{1-zS_2(z)} = \underbrace{\frac{1}{3}\frac{1+z}{1-z}}_{\mathcal{R}_s(z)} + \underbrace{\frac{1}{6}\frac{1-0.5z}{1+0.5z}}_{\mathcal{R}_a(z)}\\ &= \frac{1}{2} + \frac{1}{2}z + \frac{3}{4}z^2 + \cdots \quad |z| < 1\end{aligned}$$

in which the singular and absolutely continuous parts are distinguished. From the series expansion of $\mathcal{R}(z)$, we identify $r_1 = \frac{1}{2}$, $r_2 = \frac{3}{4}$, and so on. Similarly

$$U(z) = \sqrt{r_0}\frac{zS_1(z)}{1-zS_2(z)} = \frac{1}{2}\frac{z(z-1)}{(1+0.5z)(1-z)} = \frac{-0.5z}{1+0.5z}$$

whose series expansion yields $q_1 = u_{-1} = -\frac{1}{2}$ and $q_2 = u_{-2} = \frac{1}{4}$. The matrix $\mathbf{P}(-1)$ built from these in (13) is then positive semi-definite of rank 2 $[= \deg \mathbf{S}(z)]$. If we let $\hat{\mathcal{R}}(z)$ be the anticausal part of $U(z)U(z^{-1})$, some calculations will show that

$$\hat{\mathcal{R}}(z) = \frac{1}{6}\frac{1-0.5z}{1+0.5z} = \mathcal{R}_a(z)$$

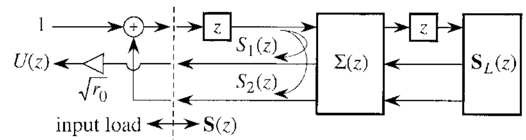

Fig. 1. Realization of $\mathbf{S}(z)$solving Problem 2 and realization of $U(z)$ solving Problem 1. Curved arrows indicate partial transfer functions $S_1(z)$ and $S_2(z)$.

so that $\mathcal{R}(z) - \hat{\mathcal{R}}(z) = \mathcal{R}_s(z)$ reduces to the singular part of $\mathcal{R}(z)$, corresponding to the positive semi-definite Toeplitz matrix

$$\begin{bmatrix} r_0 & r_1 & r_2 \\ r_1 & r_0 & r_1 \\ r_2 & r_1 & r_0\end{bmatrix} - \begin{bmatrix} \hat{r}_0 & \hat{r}_1 & \hat{r}_2 \\ \hat{r}_1 & \hat{r}_0 & \hat{r}_1 \\ \hat{r}_2 & \hat{r}_1 & \hat{r}_0\end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1\end{bmatrix}.$$

Note that $U(z)$ is the correct spectral factor of the absolutely continuous part of $\mathcal{R}(z^{-1}) + \mathcal{R}(z)$ but fails to factor the singular part because the singular part is not factorable. This example illustrates the necessity of condition (20).

IV. CONSTRUCTING $\mathbf{S}(z)$

In this section, we first present a standard procedure for constructing and parametrizing all solutions $\mathbf{S}(z)$ to Problem 2. We then show how the solution may be expressed directly in terms of the state variables of a fast least-squares algorithms, in particular, the order-recursive parameters reviewed in Section II-A2.

A. Classic Solution to Problem 2

A basic result of interpolation theory [16]–[19] is that the constraints placed on a Schur function $\mathbf{S}(z)$ solving Problem 2 can be captured using lossless multipair extraction. Fig. 1 illustrates the overall realization to be developed. To the right of the vertical dashed line, a lossless (anticausal) system $\Sigma(z)$ will be constructed such that upon closing the right-hand port with a Schur function $\mathbf{S}_L(z)$, the resulting transfer vector $\mathbf{S}(z)$ solves Problem 2. As $\mathbf{S}_L(z)$ is varied over all 2×1 Schur functions, the resulting $\mathbf{S}(z)$ will vary over all Schur functions solving Problem 2. Once such a Schur function $\mathbf{S}(z)$ is constructed, a realization of $U(z)$ per (21) is readily obtained by closing one port to the left of the dashed line in Fig. 1 and scaling the remaining output by $\sqrt{r_0}$.

We review now how to construct $\Sigma(z)$; for the benefit of the nonexpert, a brief verification that the construct works is included. More detail on this and related interpolation problems may be found in [16] and [17].

We begin with the data array

$$\mathbf{G} = \begin{bmatrix} \sqrt{r_0} & r_1/\sqrt{r_0} & r_2/\sqrt{r_0} & \cdots & r_M/\sqrt{r_0} \\ 0 & q_1 & q_2 & \cdots & q_M \\ 0 & r_1/\sqrt{r_0} & r_2/\sqrt{r_0} & \cdots & r_M/\sqrt{r_0}\end{bmatrix} \tag{26}$$

whose three rows contain the leading terms of the series expansions of $a(z)$, $c(z)$, and $b(z)$ from (15), (18), and (17), respectively.

1) Shift the first row of the array (26) one position to the right:

$$(26) \xrightarrow{z} \begin{bmatrix} 0 & \sqrt{r_0} & r_1/\sqrt{r_0} & \cdots & r_{M-1}/\sqrt{r_0} \\ 0 & q_1 & q_2 & \cdots & q_M \\ 0 & r_1/\sqrt{r_0} & r_2/\sqrt{r_0} & \cdots & r_M/\sqrt{r_0} \end{bmatrix}.$$

2) Choose a hyperbolic rotation to annihilate the second element of the first nonzero column. In the first pass, this appears as

$$\begin{bmatrix} 1/\cos\theta_0 & \sin\theta_0/\cos\theta_0 & 0 \\ \sin\theta_0/\cos\theta_0 & 1/\cos\theta_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & \sqrt{r_0} & r_1/\sqrt{r_0} & \cdots & r_{M-1}/\sqrt{r_0} \\ 0 & q_1 & q_2 & \cdots & q_M \\ 0 & r_1/\sqrt{r_0} & r_2/\sqrt{r_0} & \cdots & r_M/\sqrt{r_0} \end{bmatrix} = \begin{bmatrix} 0 & y_1 & \times & \cdots & \times \\ 0 & 0 & \times & \cdots & \times \\ 0 & r_1/\sqrt{r_0} & r_2/\sqrt{r_0} & \cdots & r_M/\sqrt{r_0} \end{bmatrix}$$

in which $y_1 = \sqrt{r_0 - q_1^2}$, and $\sin\theta_0 = -q_1/\sqrt{r_0}$.

3) Choose a hyperbolic rotation to annihilate the third element of the first nonzero column. In the first pass, this appears as

$$\begin{bmatrix} 1/\cos\phi_0 & 0 & \sin\phi_0/\cos\phi_0 \\ 0 & 1 & 0 \\ \sin\phi_0/\cos\phi_0 & 0 & 1/\cos\phi_0 \end{bmatrix} \times \begin{bmatrix} 0 & y_1 & \times & \cdots & \times \\ 0 & 0 & \times & \cdots & \times \\ 0 & r_1/\sqrt{r_0} & r_2/\sqrt{r_0} & \cdots & r_M/\sqrt{r_0} \end{bmatrix} = \begin{bmatrix} 0 & y_2 & \times & \cdots & \times \\ 0 & 0 & \times & \cdots & \times \\ 0 & 0 & \times & \cdots & \times \end{bmatrix} \quad (27)$$

in which $y_2 = \sqrt{y_1^2 - r_1^2/r_0}$ and $\sin\phi_0 = -(r_1/\sqrt{r_0})/y_1$.

4) Replace the array (26) with (27), and reiterate the above $M-1$ times to eliminate all the elements of the second and third rows.

This procedure is a Schur algorithm and continues M full iterations with

$$|\sin\theta_k| < 1 \quad \text{and} \quad |\sin\phi_k| < 1, \quad \text{for all } k$$

if and only if $\mathbf{P}(-1)$ from (10) is positive definite (e.g., [23]). If $\mathbf{P}(-1)$ is positive semi-definite of rank $m < M+1$, the procedure terminates after m stages and yields

$$|\sin\theta_{m-1}| = 1 \quad \text{or} \quad |\sin\phi_{m-1}| = 1.$$

Now, let $a(z)$, $c(z)$, and $b(z)$ be three anticausal functions whose leading terms agree with those of (15), (18), and (17), respectively. The shift-and-rotate operations on the leading coefficients of these functions appears in Fig. 2 for the case $M = 3$. (For larger M, simply cascade further sections). At the right end of the figure, the resulting function $a_M(z)$ has M leading zeros in its series expansion, whereas $c_M(z)$ and $b_M(z)$ have $M+1$ leading zeros by virtue of their leading terms being annihilated.

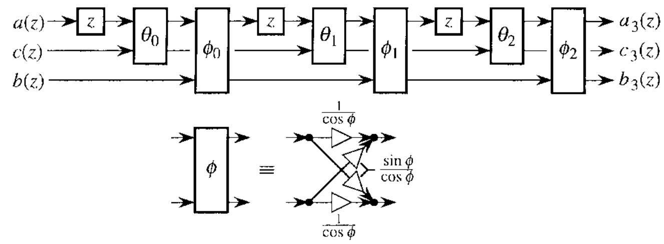

Fig. 2. Illustrating the Schur recursion as shift and rotate operations on coefficients of functions.

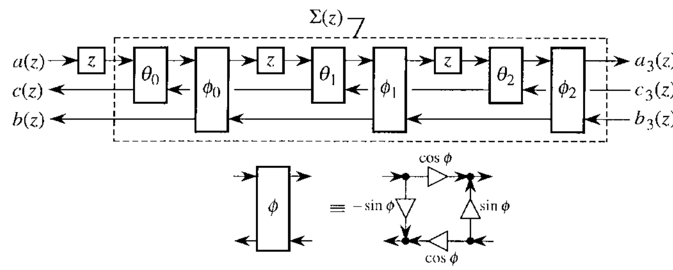

Fig. 3. Redrawing of Fig. 2, yielding lossless transfer matrix $\Sigma(z)$.

Consider now reversing the flow direction of the lower two branches of Fig. 2, giving Fig. 3, and designate the transfer matrix by $\Sigma(z)$

$$\begin{bmatrix} a_M(z) \\ c(z) \\ b(z) \end{bmatrix} = \Sigma(z) \begin{bmatrix} za(z) \\ b_M(z) \\ c_M(z) \end{bmatrix}.$$

Each hyperbolic rotation of Fig. 2 is converted to a plane rotation in Fig. 3, but the relations between the intermediate signals in the two figures are the same. As such, suppose $a(z)$ is any anticausal function whose M leading terms agree with those of (15), and let $c_M(z)$ and $b_M(z)$ be any two anticausal functions whose $M+1$ leading terms vanish. The resulting outputs $c(z)$ and $b(z)$ must then have leading terms in their expansions agree with those of (18) and (17), respectively, and similarly, the resulting output $a_M(z)$ has M leading zeros in its series expansion.

Now, since $\Sigma(z)$ is the interconnection of plane rotations and advance operations, both of which preserve L_2 norms, $\Sigma(z)$ is lossless $\Sigma^t(z^{-1})\Sigma(z) = \mathbf{I}$ for all z. Now close the right-hand port of $\Sigma(z)$ according to

$$\begin{bmatrix} c_M(z) \\ b_M(z) \end{bmatrix} = \mathbf{S}_L(z) z a_M(z).$$

By a basic result of network synthesis of scattering functions (e.g., [16], [18], [19], [21]), the resulting function $\mathbf{S}(z)$ in Fig. 1 will be a Schur (resp. lossless) function if and only if $\mathbf{S}_L(z)$ is a Schur (resp. lossless) function. Observe finally that Problem 2 constrains only the first $M+1$ terms of the anticausal impulse response of $\mathbf{S}(z)$; these terms are insensitive to the choice of $\mathbf{S}_L(z)$ on observing the advance operators "z" in the upper branch of Figs. 1 and 3.

This verifies that Fig. 1 provides a family of solutions $\mathbf{S}(z)$ to Problem 2. That all such solutions may be so generated—using the same $\Sigma(z)$ matrix—can likewise be shown; the interested reader may find complete treatments in [16], [17].

B. Relation to Fast Least-Squares Parameters

As the parameters $\{r_k\}$ and $\{q_k\}$ are not those commonly encountered in fast least-squares algorithms, it is useful to express the rotation angles of Fig. 3 in terms of the state variables reviewed in Section II-A.

Identity 4: The rotation angles $\{\theta_k\}$ and $\{\phi_k\}$ of the order recursion (8) are precisely the angles determined from the above Schur algorithm using $\{r_k\}$ and $\{q_k\}$.

The basic idea of this identity is to note that the Schur algorithm of Section IV-A operates on the generate vectors of $\mathbf{P}$, whereas (8) operates on the generator vectors of $\left[\begin{smallmatrix}\mathbf{P}^{-1} & \mathbf{0}\\ \mathbf{0}^t & 0\end{smallmatrix}\right]$. The details are presented in Appendix A.

These rotation angles make an explicit appearance in, e.g., the fast QR algorithm studied in [3][3] and can readily be inferred from other minimal lattice and fast QR algorithms (e.g., [5], [9], [10]). Fast transversal filters, by contrast, do not always allow reconstruction of the past input, as backward consistency is readily violated in such algorithms; this issue is examined in some detail in [1] and [2].

Having now the rotation angles that build $\Sigma(z)$, we must specify a lossless load $\mathbf{S}_L(z)$ in Fig. 1 for which the constraint (20) on $zS_2(z)$ is satisfied. Appendix B shows that whenever $\mathbf{P}(-1)$ is positive definite, such a lossless load may be found and outlines a procedure for determining a simple choice of the form $\mathbf{S}_L = \left[\begin{smallmatrix}\sin\alpha\\ \cos\alpha\end{smallmatrix}\right]$. When $\mathbf{P}(-1)$ is positive semi-definite (and singular), by constrast, the solution set to Problem 2 reduces to a sole lossless $\mathbf{S}(z)$ for which (20) may or may not be satisfied, depending on the values $\{r_k\}$ and $\{q_k\}$. The values for $\{r_k\}$ and $\{q_k\}$ for which Problem 1 may be solved thus defines an open set.

V. Concluding Remarks

Our main result is to show how to reconstruct the set of valid past inputs to a fast least-squares filtering algorithm given the internal variables at a given time instant. Although algebraically equivalent to the system theory problems tackled in [6]–[8], our formulation reveals a supplementary condition, as in [12] and [13], required to reach correct necessary and sufficient conditions for solvability. Our construct is also order recursive, in contrast to the solutions of [6]–[8], where this property has resulted from connections made with recursive solutions to interpolation problems or, equivalently, the well-known order recursive nature of lattice and fast QR algorithms.

Our results are expected to have implications in studying accuracy aspects in fast least-squares algorithms. The ability to reconstruct a valid past input (when applicable) fits into the "strong stability" classification of Bunch [24], in the sense that the given covariance matrix $\mathbf{P}$ associated with the perturbed parameters may still be factored into a data matrix displaying the correct shift structure. Our formulas for past input reconstruction should allow one to study how much a past input is disturbed on absorbing numerical perturbations into the state variables of a given fast least-squares algorithm. A meaningful basis for judging accuracy is to compare the relative magnitudes of past input perturbations among different algorithms and favoring that for which the past input perturbation is smallest. Further study is required in this direction.

[3] The angles θ_k in Fig. 3 are precisely those of [3], but the angles ϕ_k are denoted by ϕ_{k+1} in [3]. The index on ϕ is decremented in this paper so that rotations within a common section of Fig. 3 take the same index.

Appendix A
Identity 4

We use results showing how Schur reduction applied to the generator vectors of a structured matrix can be used to construct generator vectors corresponding to the inverse of the matrix [25].

Consider the initial generator functions

$$\begin{aligned} a_0(z) &= \sqrt{r_0} + \frac{r_1}{\sqrt{r_0}}z + \cdots + \frac{r_M}{\sqrt{r_0}}z^M \\ b_0(z) &= 0 + \frac{r_1}{\sqrt{r_0}}z + \cdots + \frac{r_M}{\sqrt{r_0}}z^M \\ c_0(z) &= 0 + q_1 z + \cdots + q_M z^M \end{aligned} \tag{28}$$

obtained from the generator vectors of $\mathbf{P}$ in (14). The Schur procedure of Section IV then appears as

$$\begin{bmatrix} a_{k+1}(z)\\ b_{k+1}(z)\\ c_{k+1}(z)\end{bmatrix} = \begin{bmatrix} \frac{1}{\cos\phi_k} & \frac{\sin\phi_k}{\cos\phi_k} & 0\\ \frac{\sin\phi_k}{\cos\phi_k} & \frac{1}{\cos\phi_k} & 0\\ 0 & 0 & 1\end{bmatrix} \times \begin{bmatrix} \frac{1}{\cos\theta_k} & 0 & \frac{\sin\theta_k}{\cos\theta_k}\\ 0 & 1 & 0\\ \frac{\sin\theta_k}{\cos\theta_k} & 0 & \frac{1}{\cos\theta_k}\end{bmatrix} \begin{bmatrix} za_k(z)\\ b_k(z)\\ c_k(z)\end{bmatrix} \tag{29}$$

where the angles θ_k and ϕ_k are chosen to annihilate the kth term in the series expansions of $c_k(z)$ and $b_k(z)$, respectively.

Now, a set of generator functions is said to be "admissible" [23] if there exist constants μ_1, μ_2, and μ_3 such that $\mu_1 a_0(z) - \mu_2 b_0(z) - \mu_3 c_0(z) = 1$. The set in (28) is admissible, using $\mu_1 = \mu_2 = 1/\sqrt{r_0}$, $\mu_3 = 0$. By an interesting result [25], [15, Sec. 5], if we initialize

$$E_0(z) = \mu_1, \quad F_0(z) = \mu_2, \quad G_0(z) = \mu_3$$

and then apply the order recursion from (29), viz

$$\begin{bmatrix} E_{k+1}(z)\\ F_{k+1}(z)\\ G_{k+1}(z)\end{bmatrix} = \begin{bmatrix} \frac{1}{\cos\phi_k} & \frac{\sin\phi_k}{\cos\phi_k} & 0\\ \frac{\sin\phi_k}{\cos\phi_k} & \frac{1}{\cos\phi_k} & 0\\ 0 & 0 & 1\end{bmatrix} \times \begin{bmatrix} \frac{1}{\cos\theta_k} & 0 & \frac{\sin\theta_k}{\cos\theta_k}\\ 0 & 1 & 0\\ \frac{\sin\theta_k}{\cos\theta_k} & 0 & \frac{1}{\cos\theta_k}\end{bmatrix} \begin{bmatrix} zE_k(z)\\ F_k(z)\\ G_k(z)\end{bmatrix} \tag{30}$$

then $E_M(z)$, $F_M(z)$, and $G_M(z)$ will yield generator functions from the displacement structure of the inverse $\mathbf{P}^{-1}$. Observe now that if we rename the functions as

$$E_k(z) = B_k(z), \quad F_k(z) = A_k(z), \quad G_k(z) = C_k(z)$$

then (30) concides with (8). However, by construction, $A_k(z)$, $B_k(z)$, and $C_k(z)$ are already generator functions obtained from displacing the inverse $\mathbf{P}^{-1}$ in (7). It follows that the rotation angles $\{\theta_k\}$ and $\{\phi_k\}$ occuring in the order recursion (8) are the same as those determined from the Schur procedure of (29), thereby giving Identity 4.

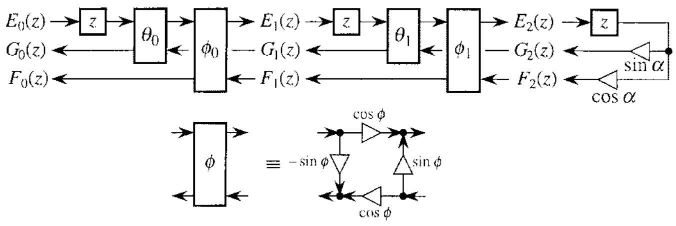

Fig. 4. Realization of $\mathbf{S}(z)$upon choosing a simple lossless $\mathbf{S}_L(z)$.

Appendix B
Choosing $\mathbf{S}_L(z)$

We show here that when $\mathbf{P}(-1)$ is positive definite, one can always choose a lossless $\mathbf{S}_L(z)$ in Fig. 3 such that the resulting $zS_2(z)$ obeys the condition (20).

Consider the simplest lossless choice for $\mathbf{S}_L(z)$, viz

$$\mathbf{S}_L(z) = \begin{bmatrix} \sin\alpha \\ \cos\alpha \end{bmatrix}$$

where α may be chosen freely. The realization of Fig. 3 would then appear, for $M = 2$, as in Fig. 4 (with straightforward extension to larger M). Let $E_0(z)$ be the z-transform of any (anticausal) input sequence, and let $G_0(z)$ and $F_0(z)$ be the z transforms of the two output sequences obtained for a given choice of α. We then have

$$\begin{bmatrix} G_0(z) \\ F_0(z) \end{bmatrix} = \begin{bmatrix} zS_1(z) \\ zS_2(z) \end{bmatrix} E_0(z). \tag{31}$$

Let $[E_k(z), G_k(z), F_k(z)]$ be the z transforms of the intermediate signals appearing in Fig. 4; these are related as

$$\begin{bmatrix} E_{k+1}(z) \\ G_k(z) \\ F_k(z) \end{bmatrix} = \begin{bmatrix} \cos\theta_k & \sin\theta_k & 0 \\ -\sin\theta_k & \cos\theta_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\phi_k & 0 & \sin\phi_k \\ 0 & 1 & 0 \\ -\sin\phi_k & 0 & \cos\phi_k \end{bmatrix} \begin{bmatrix} zE_k(z) \\ G_{k+1}(z) \\ F_{k+1}(z) \end{bmatrix}, \quad k = 0, 1, \ldots, M-1 \tag{32}$$

with the boundary condition

$$\begin{bmatrix} G_M(z) \\ F_M(z) \end{bmatrix} = \begin{bmatrix} \sin\alpha \\ \cos\alpha \end{bmatrix} zE_M(z). \tag{33}$$

We assume first that $\mathbf{P}(-1)$ is positive definite such that the rotation angles $\{\theta_k\}$ and $\{\phi_k\}$ from the Schur algorithm of Section IV fulfill $|\sin\theta_k| < 1$ and $|\sin\phi_k| < 1$. The relation (32) can then be rearranged as

$$\begin{bmatrix} E_{k+1}(z) \\ G_{k+1}(z) \\ F_{k+1}(z) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\cos\phi_k} & 0 & \frac{\sin\phi_k}{\cos\phi_k} \\ 0 & 1 & 0 \\ \frac{\sin\phi_k}{\cos\phi_k} & 0 & \frac{1}{\cos\phi_k} \end{bmatrix} \begin{bmatrix} \frac{1}{\cos\theta_k} & \frac{\sin\theta_k}{\cos\theta_k} & 0 \\ \frac{\sin\theta_k}{\cos\theta_k} & \frac{1}{\cos\theta_k} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{\Phi_k(z)} \times \begin{bmatrix} E_k(z) \\ G_k(z) \\ F_k(z) \end{bmatrix}. \tag{34}$$

Suppose now that for some $z = e^{j\omega}$, we obtain $e^{j\omega}S_2(e^{j\omega}) = 1$, thereby violating (20); this then implies $S_1(e^{j\omega}) = 0$ by way of (19). We may then trace back, using (34) and (33), what constraint must result on α.

Now, for this value of ω, we must have in (31)

$$\begin{bmatrix} G_0(e^{j\omega}) \\ F_0(e^{j\omega}) \end{bmatrix} = \begin{bmatrix} e^{j\omega}S_1(e^{j\omega}) \\ e^{j\omega}S_2(e^{j\omega}) \end{bmatrix} E_0(e^{j\omega}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} E_0(e^{j\omega})$$

or

$$\begin{bmatrix} E_0(e^{j\omega}) \\ G_0(e^{j\omega}) \\ F_0(e^{j\omega}) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} E_0(e^{j\omega}). \tag{35}$$

Insert this now into the recursion (34), and observe that the complex amplitude $E_0(e^{j\omega})$ simply contributes a scalar multiple to this system; without loss of generality, we suppose $E_0(e^{j\omega}) = 1$. The recursion (34) then appears as

$$\begin{bmatrix} E_M(z) \\ G_M(z) \\ F_M(z) \end{bmatrix} = \Phi_{M-1}(z) \cdots \Phi_0(z) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

where the three polynomials $E_M(z)$, $G_M(z)$, and $F_M(z)$ do not, by this formula, depend on α.

In order for the boundary condition (33) involving α to be met, we must now have, for our particular value of ω

$$\frac{G_M(e^{j\omega})}{e^{j\omega}E_M(e^{j\omega})} = \sin\alpha, \quad \frac{F_M(e^{j\omega})}{e^{j\omega}E_M(e^{j\omega})} = \cos\alpha. \tag{36}$$

Observe that although $\sin\alpha$ and $\cos\alpha$ are both real valued, the left-hand sides will, in general, be complex valued. This implies that (35) can hold only for those ω for which the imaginary parts of the left-hand sides of (36) both vanish. Accordingly, let $\{\omega_k\}$ denote this set. (Since the polynomials all have real coefficients, this set will always include $\omega = 0$ and $\omega = \pi$ and possibly other values of ω as well.) The set $\{\omega_k\}$ is comprised of distinct values since if the imaginary parts of the left-hand sides of (36) were to vanish for some continuum of points on the unit circle $|z| = 1$, analyticity conditions would be violated. Collect now the ordered pairs of real values

$$\left[\frac{G_M(e^{j\omega_k})}{e^{j\omega_k}E_M(e^{j\omega_k})}, \frac{F_M(e^{j\omega_k})}{e^{j\omega_k}E_M(e^{j\omega_k})}\right].$$

If we choose α such that $[\sin\alpha, \cos\alpha]$ agrees with one of these ordered pairs, then (35) is verified for the ω_k in question, and we force the condition $e^{j\omega_k}S_2(e^{j\omega_k}) = 1$. Conversely, choosing $[\sin\alpha, \cos\alpha]$ to agree with none of these ordered pairs must then contradict (35), giving $e^{j\omega}S_2(e^{j\omega}) \neq 1$ for all ω, which is the relation so sought. This proves that when $\mathbf{P}(-1)$ is positive definite, we may always find a lossless $\mathbf{S}_L(z)$, which leads to satisfaction of (20).

When $\mathbf{P}(-1)$ is positive semi-definite of rank $m < M+1$, we obtain $\sin\theta_{m-1} = \pm 1$ or $\sin\phi_{m-1} = \pm 1$. The relation (34) remains valid for $k < m-1$, whereas (33) is replaced by

$$\begin{bmatrix} G_{m-1}(z) \\ F_{m-1}(z) \end{bmatrix} = \begin{bmatrix} -\sin\theta_{m-1} \\ \mp\cos\theta_{m-1} \end{bmatrix} zE_{m-1}(z)$$

such that θ_{m-1} plays the role of α in the above analysis. In contrast to α, however, θ_{m-1} is fixed, and the condition (20) may or may not be satisfied.

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