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Stability of Multivariable Least-Squares Models

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***Abstract*—Least-squares equation-error models are widely used as a simple means of estimating an input-output transfer function. Although the models furnished by the least-squares method are not always stable, some recent works have shown that an autoregressive constraint on the input is sufficient to ensure stability of the furnished model. Here we provide a simple proof of this property for multivariable system estimation.**

I. INTRODUCTION

LEAST-squares (or equation-error) modeling techniques are often used as a simple means to estimate an unknown transfer function in a system identification context. Although the dynamic model furnished by this method is not always stable [1], stability does apply provided the input is restricted to a white noise [2] (scalar case), [3] (multivariable case) or, more generally, an autoregressive process [5] (scalar case). We develop here an elementary proof that an autoregressive property of the input is sufficient to ensure stability of the furnished model, thereby extending a result from [5] to the multivariable case. Our result applies to all stationary causal systems with stationary additive output disturbances. In what follows, the expectation operator is denoted as $E[\cdot]$, and z denotes the backward shift operator, i.e., $z\,\mathbf{u}(n) = \mathbf{u}(n-1)$.

Many system identification problems are based on the following q-input, p -output multivariable description of the system under study

$$\begin{aligned}\mathbf{y}(n) &= \sum_k \mathbf{G}_k\,\mathbf{u}(n-k) + \mathbf{v}(n)\\ &= \left(\sum_k \mathbf{G}_k\,z^k\right)\mathbf{u}(n) + \mathbf{v}(n)\\ &= \mathbf{G}(z)\,\mathbf{u}(n) + \mathbf{v}(n) \qquad (1)\end{aligned}$$

where $\{\mathbf{u}(\cdot)\}$ is a sequence of q-vectors and $\{\mathbf{y}(\cdot)\}$ is a sequence of p-vectors. We assume in addition that:

- The vector processes $\{\mathbf{u}(\cdot)\}$ and $\{\mathbf{y}(\cdot)\}$ are jointly stationary;
- The $p \times q$ transfer matrix $\mathbf{G}(z)$ is causal and stable, i.e.

$$\mathbf{G}(z) = \sum_{k=0}^{\infty} \mathbf{G}_k\,z^k, \qquad \text{with } \sum_{k=0}^{\infty} \|\mathbf{G}_k\|^2 < \infty;$$

- The disturbance term $\{\mathbf{v}(\cdot)\}$ is uncorrelated with the input, i.e.

$$E[\mathbf{u}(n)\,\mathbf{v}^t(m)] = \bigcirc_{p\times q}, \qquad \text{for all } m, n$$

and that the covariance matrix constructed from $\{\mathbf{v}(\cdot)\}$ has full rank.

We introduce now two matrix polynomials

$$\begin{aligned}\mathbf{A}(z) &= \mathbf{I} + \mathbf{A}_1\,z + \cdots + \mathbf{A}_M\,z^M \qquad (p\times p)\\ \mathbf{B}(z) &= \mathbf{B}_0 + \mathbf{B}_1\,z + \cdots + \mathbf{B}_M\,z^M \qquad (p\times q).\end{aligned}$$

A least-squares equation-error estimate of $\mathbf{G}(z)$ is obtained by minimizing the "residue variance"

$$\min_{\mathbf{A}(z),\,\mathbf{B}(z)} E\|\mathbf{A}(z)\,\mathbf{y}(n) - \mathbf{B}(z)\,\mathbf{u}(n)\|^2 \qquad (2)$$

where $\|\cdot\|$ denotes the Euclidean vector norm. The polynomials obtained at the minimum are denoted as $\widehat{\mathbf{A}}(z)$ and $\widehat{\mathbf{B}}(z)$. An estimate for the system $\mathbf{G}(z)$ may then be taken as

$$\widehat{\mathbf{G}}(z) = [\widehat{\mathbf{A}}(z)]^{-1}\,\widehat{\mathbf{B}}(z).$$

In many applications, the unknown system $\mathbf{G}(z)$ may have a very large, if not infinite, order. We assume that the chosen polynomial degree M does not allow an exact fit between $\widehat{\mathbf{G}}(z)$ and $\mathbf{G}(z)$, corresponding to practical reduced-order cases.

A desirable property in most applications is that the estimate $\widehat{\mathbf{G}}(z)$ be stable and causal whenever $\mathbf{G}(z)$ is. This reduces to the property that the matrix polynomial $\widehat{\mathbf{A}}(z)$ be minimum phase (or causally invertible), i.e., [6]

$$\det \widehat{\mathbf{A}}(z_0) = 0 \qquad \Rightarrow \qquad |z_0| > 1.$$

It is known [1] that $\widehat{\mathbf{A}}(z)$ obtained from the minimization problem (2) is not necessarily minimum phase, unless further constraints on the input $\{\mathbf{u}(\cdot)\}$ are imposed. For example, if $\{\mathbf{u}(\cdot)\}$ is a white noise process, the minimizing $\widehat{\mathbf{A}}(z)$ will be minimum phase (see [2] for the scalar case, and [3] and [4] for the multivariable case). This result has also been extended to autoregressive inputs for the scalar case in [5].

The present note will provide a direct proof of the minimum-phase property of $\widehat{\mathbf{A}}(z)$ for the multivariable case, under an autoregressive constraint on the input $\{\mathbf{u}(\cdot)\}$. Therefore, we assume that the input $\{\mathbf{u}(\cdot)\}$ is an autoregressive process of degree less than or equal to M, generated as

$$\mathbf{u}(n) = [\Gamma(z)]^{-1}\,\epsilon(n) \qquad (3)$$

where the matrix polynomial

$$\Gamma(z) = \mathbf{I} + \Gamma_1\,z + \cdots + \Gamma_M\,z^M \qquad (q\times q)$$

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is minimum phase,[1] and $\{\epsilon(\cdot)\}$ is a white noise process, i.e.

$$E[\epsilon(n)\,\epsilon^t(m)] = \begin{cases} \mathbf{S} > 0, & m = n; \\ \bigcirc, & m \neq n\,. \end{cases}$$

In this case, $\{\epsilon(\cdot)\}$ represents an innovation sequence corresponding to $\{\mathbf{u}(\cdot)\}$, and the orthogonality relation

$$E[\epsilon(n)\,\mathbf{u}^t(k)] = \bigcirc, \qquad \text{for all } k < n \tag{4}$$

then applies.

II. THE MAIN RESULT

Theorem: If $\{\mathbf{u}(\cdot)\}$ is an autoregressive process of degree M or less, the least-squares problem (2) furnishes $\widehat{\mathbf{A}}(z)$ minimum phase.

Proof: The proof involves embedding the given problem (2) into a vector linear prediction problem. A well known minimum phase property of vector linear prediction will then yield the theorem.

To this end, let $\mathbf{x}(n)$ be a stationary second-order vector process, and let

$$\mathbf{Q}(z) = \mathbf{Q}_0 + \mathbf{Q}_1\,z + \cdots + \mathbf{Q}_M\,z^M$$

be a square matrix polynomial. Suppose also that the leading term $\mathbf{Q}_0$ is of the form

$$\mathbf{Q}_0 = \begin{bmatrix} \mathbf{I}_p & \mathbf{Q}_{12} \\ \bigcirc & \mathbf{I}_q \end{bmatrix}. \tag{5}$$

We may then introduce the vector prediction problem

$$\min_{\mathbf{Q}(z)} E\|\mathbf{Q}(z)\,\mathbf{x}(n)\|^2$$

whose properties have been studied extensively [7]–[9].

Let

$$\mathbf{R} = E\left\{ \begin{bmatrix} \mathbf{x}(n) \\ \vdots \\ \mathbf{x}(n-M) \end{bmatrix} [\mathbf{x}^t(n) \quad \cdots \quad \mathbf{x}^t(n-M)] \right\}.$$

By construction, this covariance matrix is at least positive semi-definite. If positive definite (i.e., full rank), then the minimizing $\mathbf{Q}(z)$, denoted $\widehat{\mathbf{Q}}(z)$, is uniquely determined, and yields a minimum phase polynomial (e.g., [7]–[9]).

We now show that problem (2) may be embedded into a vector prediction problem of the above type. To this end, introduce two auxiliary matrix polynomials

$$\begin{aligned} \mathbf{C}(z) &= \mathbf{I} + \mathbf{C}_1\,z + \cdots + \mathbf{C}_M\,z^M \qquad (q \times q) \\ \mathbf{D}(z) &= \mathbf{0} + \mathbf{D}_1\,z + \cdots + \mathbf{D}_M\,z^M \qquad (q \times p). \end{aligned}$$

Note in particular that $\mathbf{D}(z)$ is strictly causal, i.e., $\mathbf{D}(0) = \bigcirc$. Next, consider the augmented problem

$$\min_{\mathbf{Q}(z)} \left\| \underbrace{\begin{bmatrix} \mathbf{A}(z) & \mathbf{B}(z) \\ \mathbf{D}(z) & \mathbf{C}(z) \end{bmatrix}}_{\triangleq\, \mathbf{Q}(z)} \begin{bmatrix} \mathbf{y}(n) \\ \mathbf{u}(n) \end{bmatrix} \right\|^2. \tag{6}$$

Observe that the leading term

$$\mathbf{Q}(0) = \begin{bmatrix} \mathbf{A}(0) & \mathbf{B}(0) \\ \mathbf{D}(0) & \mathbf{C}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{B}_0 \\ \bigcirc & \mathbf{I} \end{bmatrix}$$

is precisely of the form assumed in (5) above. Upon setting $\mathbf{x}(n) = \left[\begin{smallmatrix} \mathbf{y}(n) \\ \mathbf{u}(n) \end{smallmatrix}\right]$, and noting that the covariance matrix of this vector is positive definite under the assumptions made (see, e.g., [7]), it follows that the minimizing argument $\widehat{\mathbf{Q}}(z)$ will be unique and minimum phase.

It is easy to check that problem (6) decouples into two problems, namely the given problem (2) plus

$$\min_{\mathbf{C}(z),\,\mathbf{D}(z)} E\|\mathbf{D}(z)\,\mathbf{y}(n) + \mathbf{C}(z)\,\mathbf{u}(n)\|^2.$$

With the assumed data generating mechanisms (1), (3), and (4), we can observe the decomposition

$$\begin{aligned} &\mathbf{D}(z)\,\mathbf{y}(n) + \mathbf{C}(z)\,\mathbf{u}(n) \\ &\quad = \epsilon(n) + \Big[\mathbf{C}(z) - \Gamma(z) + \mathbf{D}(z)\,\mathbf{G}(z)\Big]\mathbf{u}(n) + \mathbf{D}(z)\,\mathbf{v}(n). \end{aligned}$$

Note that, since $\mathbf{C}(z)$ and $\Gamma(z)$ have the same leading coefficient $\mathbf{I}$, the transfer function of the second term of the right-hand side is strictly causal. The three terms on the right-hand side are thus mutually uncorrelated, so that

$$\min_{\mathbf{C}(z),\,\mathbf{D}(z)} E\|\mathbf{D}(z)\,\mathbf{y}(n) + \mathbf{C}(z)\,\mathbf{u}(n)\|^2 \geq E\|\epsilon(n)\|^2.$$

Equality is clearly attained for the optimized values given by

$$\widehat{\mathbf{D}}(z) \equiv \bigcirc, \qquad \widehat{\mathbf{C}}(z) = \Gamma(z).$$

The optimized $\widehat{\mathbf{Q}}(z)$ is thus of the form

$$\widehat{\mathbf{Q}}(z) = \begin{bmatrix} \widehat{\mathbf{A}}(z) & \widehat{\mathbf{B}}(z) \\ \bigcirc & \Gamma(z) \end{bmatrix}$$

for which

$$\det \widehat{\mathbf{Q}}(z) = \det \widehat{\mathbf{A}}(z) \cdot \det \Gamma(z).$$

The minimum phase character of $\widehat{\mathbf{Q}}(z)$ and $\Gamma(z)$ now implies the same for $\widehat{\mathbf{A}}(z)$. This proves the theorem.

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[1] Polynomials $\Gamma(z)$ of degree less than M are, of course, also accomodated upon setting the higher indexed coefficients to the zero matrix.